SOME RESULTS ON SOLVABLE FUZZY SUBGROUP OF A GROUP

Luma. N. M. Tawfiq and Ragaa. H. Shihab

Department of Mathematics, College of Education Ibn Al-Haitham, Baghdad University.

Abstract

In this paper we introduce an alternative definition of a solvable fuzzy group and study some of its properties. Many of new results are also proved, which is useful and important in fuzzy mathematics.

1. Introduction

The concept of fuzzy sets is introduced by [1]. Rosenfeld introduced the notion of a fuzzy group as early as 1971.

The technique of generating a fuzzy group (the smallest fuzzy group) containing an arbitrarily chosen fuzzy set was developed only in 1992 by [2].

Then many research study properties of fuzzy group, fuzzy subgroup of group, fuzzy coset, and fuzzy normal sub group of group. In this paper we introduce the definition of solvable fuzzy group and study some of its properties.

Now we introduce the following definitions which is necessary and needed in the next section:

Definition 1.1 [2], [3]

A mapping from a nonempty set $X$ to the interval $[0, 1]$ is called a fuzzy subset of $X$.

Next, we shall give some definitions and concepts related to fuzzy subsets of $G$.

Definition 1.2

Let $\mu, \nu$ be fuzzy subsets of $G$, if $\mu(x) \leq \nu(x)$ for every $x \in G$, then we say that $\mu$ is contained in $\nu$ (or $\nu$ contains $\mu$) and we write $\mu \subseteq \nu$ (or $\nu \supseteq \mu$).
If $\mu \subseteq v$ and $\mu \neq v$, then $\mu$ is said to be properly contained in $v$ (or $v$ properly contains $\mu$) and we write $\mu \subset v$ (or $v \supset \mu$).[4]

Note that: $\mu = v$ if and only if $\mu(x) = v(x)$ for all $x \in G$.[5]

**Definition 1.3** [4]

Let $\mu, v$ be two fuzzy subsets of $G$. Then $\mu \cup v$ and $\mu \cap v$ are fuzzy subsets as follows:

(i) $$(\mu \cup v)(x) = \max \{\mu(x), v(x)\}$$

(ii) $$(\mu \cap v)(x) = \min \{\mu(x), v(x)\}, \text{ for all } x \in G$$

Then $\mu \cup v$ and $\mu \cap v$ are called the union and intersection of $\mu$ and $v$, respectively.

**Definition 1.4** [5],[6]

For $\mu, v$ are two fuzzy subsets of $G$, we define the operation $\mu \circ v$ as follows:

$$(\mu \circ v)(x) = \sup \{\min \{\mu(a), v(b)\} | a, b \in G \text{ and } x = a \ast b\} \text{ For all } x \in G.$$  

We call $\mu \circ v$ the product of $\mu$ and $v$.

Now, we are ready to give the definition of a fuzzy subgroup of a group:

**Definition 1.5**[2], [7]

A fuzzy subset $\mu$ of a group $G$ is a fuzzy subgroup of $G$ if:

(i) $\min \{\mu(a), \mu(b)\} \leq \mu(a \ast b)$

(ii) $\mu(a^{-1}) = \mu(a)^{-1}$, for all $a, b \in G$.

**Theorem 1.6** [4]

If $\mu$ is a fuzzy subset of $G$, then $\mu$ is a fuzzy subgroup of $G$, if and only if, $\mu$ satisfies the following conditions:

(i) $\mu \circ \mu \subseteq \mu$

(ii) $\mu^{-1} = \mu$

where $\mu^{-1}(x) = \mu(x)$, $\forall x \in G$.

**Proposition 1.7** [7]

Let $\mu$ be a fuzzy group. Then $\mu(a) \leq \mu(e)$ $\forall a \in G$. 
Definition 1.8 [8]

If \( \mu \) is a fuzzy subgroup of \( G \), then \( \mu \) is said to be abelian , if \( \forall x, y \in G, \mu(x) > 0, \mu(y) > 0 \), then \( \mu(xy) = \mu(yx) \).

Definition 1.9 [9], [10]

A fuzzy subgroup \( \mu \) of \( G \) is said to be normal fuzzy subgroup if 
\[ \mu(x^* y) = \mu(y^* x) , \forall x, y \in G. \]

Definition 1.10 [11]

Let \( \lambda \) and \( \mu \) be two fuzzy subsets of \( G \). The commutator of \( \lambda \) and \( \mu \) is the fuzzy subgroup \( [\lambda, \mu] \) of \( G \) generated by the fuzzy subset \( (\lambda, \mu) \) of \( G \) which is defined as follows, for any \( x \in G \):

\[
(\lambda, \mu)(x) = \begin{cases} 
\sup \{ \lambda(a) \wedge \mu(b) \} & \text{if } x \text{ is a commutator} \\
\chi = [a, b] 
\end{cases}
\]

Now, we introduce the following theorems about the commutator of two fuzzy subsets of a group which are needed in the next section :

Theorem 1.11 [11]

If \( A, B \) are subsets of \( G \), then \( [\chi_A, \chi_B] = \chi_{[A,B]} \),

where for all \( x \in G \):

\[
\chi_A(a) = \begin{cases} 
1, \text{ if } a \in A \\
0, \text{ if } a \notin A 
\end{cases}
\]

Theorem 1.12 [11]

If \( \lambda, \mu, \beta \) and \( \delta \) are fuzzy subsets of \( G \) such that \( \lambda \subseteq \mu \) and \( \beta \subseteq \delta \), then 
\( [\lambda, \beta] \subseteq [\mu, \delta] \).

Definition 1.13 [12]

A fuzzy subgroup \( \mu \) of \( G \) is said to be normal fuzzy subgroup if 
\( \mu(x^* y) = \mu(y^* x) , \forall x, y \in G. \)


If \( \lambda, \beta \) are normal fuzzy subgroups of \( \mu \) and \( \delta \), respectively. Then \( [\lambda, \beta] \) is a normal fuzzy subgroup of \( [\mu, \delta] \).
Proposition 1.15 [3]

Let $\lambda$ be a fuzzy subgroup of a fuzzy group $\mu$, then $\lambda$ is a normal fuzzy subgroup in $\mu$ if and only if $\lambda_t$ is normal subgroup in $\mu_t$, $\forall t \in (0,1]$, where $\lambda(e) = \mu(e)$.

Now, we introduce an important concept about the fuzzy subset.

Definition 1.16[11]

Let $\lambda$ be a fuzzy subset of $G$. Then the tip of $\lambda$ is the supremum of the set $\{\lambda(x) | x \in G\}$.

Theorem 1.17[11]

Let $\lambda$ and $\mu$ be fuzzy subsets of $G$. Then the tip of $[\lambda, \mu]$ is the minimum of tip of $\lambda$ and tip of $\mu$.

Now, we are ready to define the concept of derived chain, which is of great importance in the next section:

Definition 1.18[11]

Let $\lambda$ be a fuzzy subgroup of $G$. We call the chain

$$
\alpha = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \ldots \supseteq \lambda^{(n)} \supseteq \ldots
$$

of fuzzy subgroups of $G$ the derived chain of $\lambda$.

2. Solvable Fuzzy Subgroups of A Group

In this section, we propose an alternative definition of a solvable fuzzy group and study some of its properties:

Definition 2.1

Let $\lambda$ be a fuzzy subgroup of $G$ with tip $\alpha$. If the derived chain $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \ldots \supseteq \lambda^{(n)} \supseteq \ldots$ of $\lambda$ terminates finitely to $(e)_{\alpha}$, then we call $\lambda$ a solvable fuzzy subgroup of $G$. If $k$ is the least nonnegative integer such that $\lambda^{(k)} = (e)_{\alpha}$, then we call the series $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \ldots \supseteq \lambda^{(k)} = (e)_{\alpha}$ the derived series of $\lambda$.

Now, we introduce another definition of solvable fuzzy groups.

Definition 2.2

Let $\lambda$ be a fuzzy subgroup of $G$ with tip $\alpha$. We call a series $\lambda = \lambda_0 \supseteq \lambda_1 \supseteq \ldots \supseteq \lambda_n = (e)_{\alpha}$ of fuzzy subgroups of $G$ a solvable series for $\lambda$ if for $0 \leq i < n$, $[\lambda_i, \lambda_i] \subseteq \lambda_{i+1}$. This is equivalent to saying that:

20
To prove the equivalence between definitions (2.1) and (2.2), we need firstly the following proposition.

**Proposition 2.3**

If \( \lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \ldots \supseteq \lambda^{(n)} = (e)_{\alpha} \) be a solvable series and \( \lambda = \lambda_0 \supseteq \lambda_1 \supseteq \ldots \supseteq \lambda_n = (e)_{\alpha} \) be a derived series, then:

\[
\lambda^{(i)} \subseteq \lambda_i, \quad i = 0, 1, \ldots, n.
\]

**Proof**

We show by induction that \( \lambda^{(i)} \subseteq \lambda_i, i = 0, 1, \ldots, n \). We have \( \lambda^{(0)} = \lambda = \lambda_0 \), and we have

\[
\lambda^{(i)} = [\lambda^{(i-1)}, \lambda^{(i-1)}] = [\lambda_0, \lambda_0] \subseteq \lambda_i.
\]

Therefore, the result holds for \( n = 1 \).

Now, let \( \lambda^{(i)} \subseteq \lambda_i \) for some \( i = 0, 1, \ldots, n - 1 \) then by theorem (1.12),

\[
\lambda^{(i+1)} = [\lambda^{(i)}, \lambda^{(i)}] \subseteq [\lambda_i, \lambda_i] \subseteq \lambda_{i+1}
\]

Hence, \( \lambda^{(i)} \subseteq \lambda_i \) for \( i = 0, 1, \ldots, n \).

Now, we are ready to prove the equivalence between definitions (2.1) and (2.2)

**Proposition 2.4**

Definition (2.1) \( \iff \) Definition (2.2).

**Proof**

Let \( \lambda \) be a fuzzy subgroup of \( G \) with tip \( \alpha \). First, suppose definition (2.1) is hold then the derived series of \( \lambda \) is a solvable series for \( \lambda \). That is definition (2.2) hold.

Conversely, suppose definition (2.2) is hold then \( \lambda \) have a solvable series \( \lambda = \lambda_0 \supseteq \lambda_1 \supseteq \ldots \supseteq \lambda_n = (e)_{\alpha} \) such that \( [\lambda_i, \lambda_i] \subseteq \lambda_{i+1} \) for some \( 0 \leq i < n \).

Form proposition (2.3), \( \lambda^{(i)} \subseteq \lambda_i \), for \( 0 \leq i \leq n \).

Therefore, we get \( (e)_{\alpha} \subseteq \lambda^{(n)} \subseteq \lambda_n = (e)_{\alpha} \). Consequently, \( \lambda \) is solvable.

That is, definition (2.1) is hold.

Now, we introduce a nontrivial example of a solvable fuzzy subgroup of the group \( S_4 \) (the group of all permutations on the set \( \{1, 2, 3, 4\} \)).

**Example 2.5**

Let \( D_4 = \{(1), (12)(34), (13)(24), (14)(23), (24), (1234), (1432), (13)\} \). Which is a dihedral subgroup of \( S_4 \) with center \( C = \{(1), (13)(24)\} \).
Let $\lambda$ be the fuzzy subset of $S_4$ defined by:

\[
\lambda(x) = \begin{cases} 
1 & \text{if } x \in C = \{(1), (13)(24)\} \\
\frac{1}{2} & \text{if } x \in \{(1234)\} \\
\frac{1}{4} & \text{if } x \in D_4 \setminus \{(1234)\} \\
0 & \text{if } x \in S_4 \setminus D_4
\end{cases}
\]

\[
\lambda^{(1)}(x) = \begin{cases} 
1 & \text{if } x = (1) \\
0 & \text{otherwise}, x \in S_4
\end{cases}
\]

And,

\[
\lambda^{(2)}(x) = \begin{cases} 
1 & \text{if } x = (1) \\
\frac{1}{4} & \text{if } x = (13)(24) \\
0 & \text{otherwise}, x \in S_4
\end{cases}
\]

Clearly, $\lambda$ is a fuzzy subgroup of $S_4$. The fuzzy subgroup $\lambda^{(1)}$ has the following definition:

Thus, we have $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \lambda^{(2)} = (e)_1$

Hence, $\lambda$ is a solvable fuzzy subgroup of $S_4$.

We will give the following remark:

**Remark 2.6**

1) If $G$ is a solvable group, then $1_G$ is a solvable fuzzy subgroup of $G$.

**Proof**

Let $G$ be a solvable group, then $G = G^{(0)} \supseteq G^{(1)} \supseteq \ldots \ldots \supseteq G^{(n)} = (e)$. Now:

$1_G = 1_{G^{(0)}} \supseteq 1_{G^{(1)}} \supseteq \ldots \ldots \supseteq 1_{G^{(n)}} \supseteq 1_{\{e\}} = (e)_1$
Hence, from definition (2.1), \( G \) is a solvable fuzzy subgroup of \( G \).

2) If \( H \) is a subgroup of \( G \), then for all \( n \geq 1 \), \( (\chi_H)^{(n)} = \chi_H \). [2]

Now, we can give the following example:

**Example 2.7**

\((S_3, \circ)\) is solvable group with \( S_3 \supseteq A_3 \supseteq \{e\} \). Now, let \( \lambda(x) = 1_{S_3} \) that is \( \lambda(x) = 1 \) for all \( x \in S_3 \):

\[
\chi^{(1)}(x) = 1_{A_3}
\]

that is \( \chi^{(1)}(x) = \begin{cases} 1 & \text{if } x \in A_3 \\ 0 & \text{otherwise} \end{cases} \)

and

\[
\chi^{(2)}(x) = 1_{\{e\}}
\]

that is \( \chi^{(2)}(x) = \begin{cases} 1 & \text{if } x = \{e\} \\ 0 & \text{otherwise} \end{cases} \)

then \( \lambda = \chi^{(0)} \supseteq \chi^{(1)} \supseteq \chi^{(2)} = (e) \)

That is \( 1_{S_3} \supseteq 1_{A_3} \supseteq \{e\} = (e) \)

Thus, \( 1_{S_3} \) is a solvable fuzzy subgroup of \( S_3 \).

Now we introduce the following theorem:

**Theorem 2.8**

A subgroup \( A \) of \( G \) is solvable if and only if \( \chi_A \) is a solvable fuzzy subgroup of \( G \)

**Proof**

Let \( A \) be any subgroup of \( G \). Consider the derived chain of \( A \)

\[
A = A^{(0)} \supseteq A^{(1)} \supseteq ........... \supseteq A^{(n)} \supseteq ...........
\]

From Remark (2.6), \( (\chi_A)^{(n)} = \chi_A \) for each \( n \geq 0 \) and therefore, \( (\chi_A)^{(n)} = (e) \) if and only if, \( A^{(n)} = (e) \). The result now is clear.

Now, we have the following result:

**Theorem 2.9**

If \( \lambda \) and \( \mu \) are two solvable fuzzy subgroups of \( G \). Then \([\lambda, \mu]\) is a solvable fuzzy subgroup of \( G \).

**Proof**

Let \( \lambda \) is solvable fuzzy subgroup of \( G \) with tip \( \alpha_1 \), then:

\[
\lambda = \chi^{(0)} \supseteq \chi^{(1)} \supseteq ........... \supseteq \chi^{(n)} = (e)_{\alpha_1}
\]

Also, \( \mu \) is solvable fuzzy subgroup of \( G \) with tip \( \alpha_2 \), then:
\[ \mu = \mu^{(0)} \supseteq \mu^{(1)} \supseteq \ldots \supseteq \mu^{(n)} = (e)\alpha_2 \]

From theorem (1.17), the tip of \([\lambda, \mu] = \alpha_1 \cap \alpha_2\). Therefore,
\[ [\lambda, \mu] = [\lambda^{(0)}, \mu^{(0)}] \supseteq [\lambda^{(1)}, \mu^{(1)}] \supseteq \ldots \supseteq [\lambda^{(n)}, \mu^{(n)}] = (e)\alpha_1 \cap \alpha_2 \]

Then \([\lambda, \mu]\) is solvable fuzzy subgroup of \(G\).

Next, we shall state the following theorem:

**Theorem 2.10**

Let \(\lambda\) be a solvable fuzzy subgroup of \(G\). Then \(\lambda_t\) is a solvable, \(\forall t \in (0, 1]\).

**Proof**

Suppose \(\lambda\) is a solvable fuzzy subgroup of \(G\) with tip \(\alpha\). Then \(\lambda\) has a solvable series
\[ \lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \ldots \supseteq \lambda^{(n)} = (e)\alpha \]

Now,
\[ \lambda_t = (\lambda^{(0)}_t) \supseteq (\lambda^{(1)}_t) \supseteq \ldots \supseteq (\lambda^{(n)}_t) = ((e)\alpha)_t = \{e\} \]

Hence, \(\lambda_t\) is a solvable subgroup of \(G\).

Now, we will give the following interesting theorem:

**Theorem 2.11**

Every fuzzy subgroup of a solvable group is solvable.

**Proof**

Let \(G\) be a solvable group. Then:
\[ G = G^{(0)} \supseteq G^{(1)} \supseteq \ldots \supseteq G^{(n)} = (e) \]

Let \(\lambda\) be any fuzzy subgroup of \(G\) with tip \(\alpha\) for \(0 \leq i \leq n\). Define \(\lambda_i\) by
\[ \lambda_i(x) = \begin{cases} \lambda(x) & \text{if } x \in G^{(i)} \\ 0 & \text{otherwise} \end{cases} \]

Then \(\lambda = \lambda_0 \supseteq \lambda_1 \supseteq \ldots \supseteq \lambda_n\) is a finite chain of fuzzy subgroups of \(G\), such that \(S(\lambda_i) \subseteq G^{(i)}\) for each \(i\), clearly \(\lambda_n = (e)\alpha\)

Let \(a, b \in G\) and \(0 \leq i \leq n-1\), if \(a \notin G^{(i)}\) or \(b \notin G^{(i)}\) then:
\[ \min \{\lambda_i(a), \lambda_i(b)\} = 0 \leq \lambda_{i+1}(a, b) \]

on the other hand, if \(a, b \in G^{(i)}\), then
\[ [a, b] \in G^{(i+1)} \]

And therefore, \(\min \{\lambda_i(a), \lambda_i(b)\} = \min \{\lambda(a), \lambda(b)\}\)
Thus, 
\[ \lambda = \lambda_0 \supseteq \lambda_1 \supseteq \ldots \supseteq \lambda_n = (e) \alpha \]

is a solvable series for \( \lambda \), and \( \lambda \) is solvable.

Now, we can obtain the following result:

**Corollary 2.12**

If \( \lambda, \mu \) are fuzzy subgroups of a solvable group \( G \), then \([\lambda, \mu]\) is solvable

**Proof**

\( \lambda, \mu \) are solvable fuzzy subgroups of \( G \) from theorem (2.11).

Then \([\lambda, \mu]\) is solvable fuzzy subgroups of \( G \) from theorem (2.9).

**Theorem 2.13**

If \( \lambda, \mu \) are fuzzy subsets of a solvable group \( G \). Then \([\lambda, \mu]\) is solvable.

**Proof**

Let \( G \) be a solvable group, then:

\[ G = G^{(0)} \supseteq G^{(1)} \supseteq \ldots \supseteq G^{(n)} = (e). \]

Now, if \( \gamma, \beta \) are the tips of \( \lambda, \mu \), respectively, then from theorem (1.17) \( \alpha = \gamma \wedge \beta \) is the tip of \([\lambda, \mu]\).

Define \([\lambda_i, \mu_i]\) as follows:

\[ [\lambda_i, \mu_i](x) = \begin{cases} [\lambda, \mu] (x) & \text{if } x \in G^{(i)} \\ \text{otherwise} & \end{cases} \]

Then \([\lambda, \mu] = [\lambda_0, \mu_0] \supseteq [\lambda_1, \mu_1] \supseteq \ldots \supseteq [\lambda_n, \mu_n] \) is a finite chain of fuzzy subgroups of \( G \), such that \( S([\lambda_i, \mu_i]) \subseteq G^{(i)} \) for each \( i \).

Clearly \([\lambda_n, \mu_n] = (e) \alpha \).
Let $a, b \in G$ and $0 \leq i \leq n - 1$, If $a \not\in G^{(i)}$ or $b \not\in G^{(i)}$, then 
$$\min\{[\lambda_i, \mu_i](a), [\lambda_i, \mu_i](b)\} = 0 \leq [\lambda_{i+1}, \mu_{i+1}](a, b)$$
on the other hand, if $a, b \in G^{(i)}$, then $[a, b] \in G^{(i+1)}$, and therefore,
$$\min\{[\lambda_i, \mu_i](a), [\lambda_i, \mu_i](b)\} = \min\{[\lambda, \mu](a), [\lambda, \mu](b)\} \leq [\lambda_{i+1}, \mu_{i+1}](a, b)$$
Thus,
$$[\lambda, \mu] = [\lambda_0, \mu_0] \supseteq [\lambda_1, \mu_1] \supseteq \ldots \supseteq [\lambda_n, \mu_n] = (e)_\alpha$$
is a solvable series for $[\lambda, \mu]$, and $[\lambda, \mu]$ is solvable.

Now, we shall state and prove the following theorem:

**Theorem 2.14**

Every fuzzy subgroup of solvable fuzzy group is solvable.

Proof

Let $\lambda$ be a solvable fuzzy subgroup of a group $G$ with tip $\alpha$ and let $\mu$ be a fuzzy
subgroup of $G$, such that $\mu \subseteq \lambda$ in view. By theorem (1.12), we get $\mu^{(n)} \subseteq \lambda^{(n)}$ for all
$n \geq 0$. Also, $\lambda$ is a solvable then $\lambda^{(n)} = (e)_\alpha$.

Thus we have $(e)_\alpha \subseteq \mu^{(n)} \subseteq \lambda^{(n)} = (e)_\alpha$, then $\mu^{(n)} = (e)_\alpha$
Hence, $\mu$ is a solvable fuzzy subgroup.

Now, we introduce the following theorem:

**Theorem 2.15**

Let $\lambda$ be a fuzzy subgroup of $G$, and $t^* = \inf\{\lambda(x)/x \in G\}$, suppose that $t^* > 0$. Then
the following are equivalent:
(i) $G$ is solvable
(ii) $\lambda_t$ is solvable
(iii) $\lambda_t$ is solvable $\forall t \in (0, 1]$.

Proof

Now, $G$ is solvable if and only if, $1_G$ is solvable, but $\lambda \subseteq 1_G$ then from theorem (2.11).

$\lambda$ is solvable then (i) $\Rightarrow$ (ii)
by theorem (2.10), (ii) $\Rightarrow$ (iii).
Now, suppose (iii) holds, \( G = \overset{\lambda}{\mu} \).

Thus \( G \) is solvable, that is \((\text{iii}) \implies (\text{i})\).

In the following proposition we obtain a sufficient condition for truth of the converse of theorem (2.14).

**Proposition 2.16**

Let \( \lambda, \mu \) be fuzzy subgroups of \( G \), such that \( S(\lambda) = S(\mu), \mu \subseteq \lambda \) and \( \mu \) is solvable. Then \( \lambda \) is solvable.

**Proof**

From theorem (2.10), \( \mu_t \) is solvable, \( \forall \ t \in (0, 1] \) 

Since \( S(\mu) \subseteq \mu_t \), then by theorem (2.14), \( S(\mu) \) is solvable and \( S(\lambda) \) is solvable.

And from theorem (2.15), \( G \) is solvable. 

Consequently, \( \lambda \) is solvable.

**Theorem 2.17**

Let \( \lambda \) and \( \mu \) be fuzzy subgroups of \( G \), such that \( \mu \) is a normal fuzzy in \( \lambda \). If \( \lambda \) is a solvable fuzzy group, then \( (\lambda/\mu)^{(t)} \) is solvable fuzzy group, \( \forall t \in (0,1] \).

**Proof**

Since \( \lambda \) is solvable fuzzy group, then from theorem (2.10), \( \lambda_t \) is solvable for all \( t \in (0,1] \).

Also, by proposition (1.15), \( \mu_t \) is normal in \( \lambda_t \) \( \forall t \in (0,1] \).

Then \( (\lambda_t/\mu_t) \) is solvable. But, \( (\lambda_t/\mu_t) \equiv (\lambda/\mu)^{(t)} \).

Then \( (\lambda/\mu)^{(t)} \) is solvable for all \( t \in (0,1] \).

Also, we have the following theorem:

**Theorem 2.18**

Let \( \lambda \) and \( \mu \) be fuzzy subgroups of \( G \), such that \( \mu \) is a normal fuzzy in \( \lambda \). If \( \lambda \) is a solvable fuzzy group, then \( (\lambda/\mu) \) is a solvable fuzzy semi-group.

**Proof**

Since \( \lambda \) is solvable fuzzy group. Then from theorem (2.10), \( \lambda_t \) is solvable for all \( t \in (0,1] \).

Also, by proposition (1.15), \( \mu_t \) is normal fuzzy subgroup in \( \lambda_t \) for all \( t \in (0,1] \).

Then \( (\lambda_t/\mu_t) \) is solvable, But \( (\lambda_t/\mu_t) \equiv (\lambda/\mu)_t \).

Hence \( (\lambda/\mu) \) is solvable fuzzy semi-group.

Next, we introduce some important propositions:
Proposition 2.19

Let $\lambda$, $\mu$ and $\gamma$ be fuzzy subgroups of $G$ such that $\gamma$ is normal fuzzy in $\lambda \text{ and } \mu$. If $\lambda$ and $\mu$ are solvable fuzzy then $[\lambda/\gamma, \mu/\gamma]$ is solvable.

Proof

Since $\lambda$, $\mu$ are solvable fuzzy subgroups of $G$ and $\gamma$ normal in $\lambda \text{ and } \mu$, then from theorem (2.18), $\lambda/\gamma$ and $\mu/\gamma$ are solvable fuzzy semi-group. Also, by theorem (2.9), $[\lambda/\gamma, \mu/\gamma]$ is solvable.

Proposition 2.20

Let $\alpha$ be a normal fuzzy subgroup of $\beta$ and $\beta$ be a normal fuzzy subgroup of $\mu$. If $\lambda$, $\mu$ are solvable fuzzy subgroups of $G$, then $([\alpha, \beta] \cap [\beta, \alpha])$ is solvable fuzzy subgroup of $G$.

Proof

Let $\lambda$, $\mu$ two solvable fuzzy subgroups of $G$, then from theorem (2.9), $[\lambda, \mu]$ is solvable fuzzy subgroup of $G$. And since $\beta, \alpha$ are normal fuzzy in $\lambda, \mu$ respectively then by corollary (1.14), $[\beta, \alpha]$ is normal fuzzy in $[\lambda, \mu]$. Thus from theorem (2.18), $([\lambda, \mu] \cap [\beta, \alpha])$ is solvable fuzzy semi-group.

References

بعض النتائج حول الزمر الضبابية القابلة للحل

لمي ناجي محمد توفيق & رجاء حامذ شهاب
قسم الرياضيات – كلية التربية أبن الهيثم – جامعة بغداد

الخلاصة:

يتم البحث تعريف الزمرة الضبابية القابلة للحل بأكثر من صيغة ثم أثبتت تكافئ الصيغ المختلفة للتعريف ودراسة خيالها
وتقديم البراهين المهمة حول المفهوم.