On Almost Countably Compact of Bitopological Spaces
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Abstract

In this paper we introduce the notion of almost countably compact spaces in bitopological spaces, give new definitions, and study some of the important properties related to this concept, the relationship between almost countably compact spaces and countably compact spaces.

Key words, compactness, $p$- compactness, countably $p$- compact of bitopological space, almost countably $p$- compact of bitopological space, $p$- regular space

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Recall that a topological space $X$ is said to be compact if every open covering $\mu$ of $X$ contains a finite subcollection that also covers $X$. A space $X$ is said to be countably compact if every countable open covering of $X$ contains a finite subcollection that covers $X$. And for a $T_1$ space $X$, countable compactness is equivalent to limit point compactness, see Munkres p178-181, [1]. By the definition of the countably compact space it is clear that every compact space is countably compact. As a generalization of countable compactness, Bonanzinga, Matveev and Pareek, [2] defined a topological space $(X, \tau)$ to be almost countably compact, if for every countable open cover $\mu$ of $X$, there exists a finite subcollection $O$ of $\mu$ such that $\bigcup \{ o \in O \} = X$. Clearly, every countably compact topological space is almost countably compact, but the converse does not hold (see Examples 2.3 and 2.4), [3]. In 1963 the study of bitopological spaces started with the paper of Kelly [4], Indeed, in this fundamental paper the concept of bitopological space itself is clearly formulated as a non empty set $X$.

**1-2. Preliminaries**

topology $i$. The cardinality of a set $A$ is denoted by $|A|$. We mean by $\omega, \omega_1, c, p-$, the first infinite cardinal, the first uncountable cardinal, the cardinality of the set of all real numbers, and pairwise, respectively. For a bitopological space $(X, \tau_1, \tau_2)$ and a subset $A \subset X$, an induced topology $\tau_i(A)$ is defined as $\tau_i(A) = A \cap \tau_i, i \in \{1, 2\}$.

open if $\mu \subset \tau_1 \cup \tau_2$ and $\mu$ contains at least one non-empty member of $\tau_1$ and at least one non-empty member of $\tau_2$. [6]

space $(X, \tau_1, \tau_2)$ is defined to be pairwise compact [6].

pairwise compact which is differing from Singal and Singal definitions [8].

each countable pairwise open cover $\mu$ of $(X, \tau_1, \tau_2)$ has a finite subcover .

Throughout this paper, all spaces $(X, \tau)$, $(X, \tau_1, \tau_2)$ (or simply $X$) are always meant as topological spaces and bitopological spaces, respectively unless explicitly stated. By $i$-open set, we shall means the open set with respect to topology $\tau_i$ in $X$, open cover of $X$, means that the cover of $X$ by $i$-open sets in $X$. By $i$-int($A$) and $i$-cl($A$) we shall mean the interior and the closure of a subset $A$ of $X$ with respect to $i$.

2- Countably p-compact and almost countably p-compact

**Definition 2-1.** A cover $\mu$ of a bitopological space $(X, \tau_1, \tau_2)$ is defined to be pairwise

**Definition 2-2.** If each pairwise open cover of $(X, \tau_1, \tau_2)$ has a finite subcover, then the

With the sense of Reilly of definition 'Pairwise Lindelof bitopological spaces', [9], we introduce the definition of countably

**Definition 2-3.** A bitopological space $(X, \tau_1, \tau_2)$ is said to be countably pairwise compact if
\( \in \mathcal{O} \cap \tau_1 \), for \( i \in \{1, 2\} \). \( \in \mathcal{O} \cap \tau_1 \) \( \in \mathcal{O} \cap \tau_1 \). Directly from the definition it can be concluded the following proposition:

subcover \( \mathcal{O} \) of \( \mu \) such that \( X = \bigcup \{ \mathcal{O} \cap \tau_1 \} \), so \( \bigcup \{ \mathcal{O} \cap \tau_1 \} \) also covers \( X \), and \( \bigcup \{ \mathcal{O} \cap \tau_1 \} \subset X \), then \( \bigcup \{ \mathcal{O} \cap \tau_1 \} = X \).

exists a finite subcollection \( \mathcal{O} \) of \( \mu \) such that \( D \subset \bigcup \{ \mathcal{O} \cap \tau_1 \} \), since \( D \) is relatively countably pairwise compact in \( X \). But \( D \) is \( \mathcal{O} \)-dense in \( X \), that is \( \mathcal{O} \)-cl \( (D) = X \) for \( i = 1, 2 \) so \( X \subset \bigcup \{ \mathcal{O} \cap \tau_1 \} \), then \( X = \bigcup \{ \mathcal{O} \cap \tau_1 \} \) which completes the proof.

not be countably pairwise compact, see also example 2-4 [3].

compact, since \( R \) is a discrete closed in \( X \), and \( X \) is almost countably pairwise compact, since \( \omega \) is \( \mathcal{O} \)-dense in \( X \) and every infinite subset of \( \omega \) has a limit point in \( X \) (Recall from [10] that a subspace \( Y \) of a space \( X \) is relatively countably compact in \( X \) if every infinite subset of \( Y \) has a limit point in \( X \)), which completes the proof.

\( \tau_2 \) is said to be pairwise regular if \( \tau_1 \)-regular with respect to \( \tau_2 \) and \( \tau_2 \) is regular with respect to \( \tau_1 \) [4].

Kelly. With the manner of Kelly the following definition is equivalents of \( X \) containing \( x \), there exists an \( \mathcal{O} \)-open set \( U \) such that \( x \in U \), \( x \in \mathcal{O} \)-cl \( (U) \), \( x \) is said to be pairwise regular (or briefly, \( \mathcal{O} \)-regular) if it is both \( \mathcal{O} \)-regular and \( \mathcal{O} \)-regular or \( \mathcal{O} \)-regular. See also [5].

\( \tau_1 \cap \tau_2 \neq \emptyset \). \( \mathcal{O} \cap \tau_1 \neq \emptyset \). Reilly, 1973, [9].

space is compact, so we can conclude the following proposition:

A be any pairwise open cover of \( X \). For each \( x \in X \), there exists an \( A_x \in \mathcal{A} \) such that \( x \in A_x \) and there exists an open set \( B_x \) of \( x \) such that \( x \in B_x \cap i - \mathcal{O} \) \( B_x \subset A_x \), let \( B = \{ B_x : x \in X \} \) then \( B \) is an open cover of

Definition 2-4. A bitopological space \( (X, \tau_1, \tau_2) \) is said to be almost countably pairwise compact if for every countable pairwise open cover \( \mu \) of \( X \), there exists a finite subcollection \( \mathcal{O} \) of \( \mu \) such that \( \bigcup \{ \mathcal{O} \cap \tau_1 \} \).

Proposition 2-5. Every countably pairwise compact bitopological space \( (X, \tau_1, \tau_2) \) is almost countably pairwise compact.

Proof: Let \( (X, \tau_1, \tau_2) \) be countably pairwise compact space, so for any countable pairwise open cover \( \mu \) of \( X \), there exist a finite \( \mathcal{O} \)-dense in \( X \) if \( \mathcal{O} \)-cl \( (A) = X \) for \( i = 1, 2 \), [5].

Proposition 2-7. Let \( D \) be an \( \mathcal{O} \)-dense subset of \( X \). If \( D \) is relatively countably pairwise compact in \( X \), then \( X \) is almost countably pairwise compact.

Proof: Let \( \mu \) be any countable pairwise open cover of \( X \), let \( D \) be an \( \mathcal{O} \)-dense subspace, relatively pairwise compact in \( X \), then there

In the following example we explain that the almost countably pairwise compact space need not be pairwise compact.

Example 2-8. The Isbell-Mr'owka space it is almost countably pairwise compact, but it is not countably pairwise compact.

Proof: Let \( X = \omega \cup R \) be the Isbell-Mr'owka space \( (\tau_1, \tau_2) \), where \( R \) is a maximal almost disjoint family of infinite subsets of \( \omega \) such that \( |R| = \mathbb{C} \). Then \( X \) is not countably pairwise compact.

Definition 2-9. In a space \( (X, \tau_1, \tau_2) \), \( \tau_1 \) is said to be regular with respect to \( \tau_2 \) if for each \( x \in X \) and a \( \tau_2 \)-closed set \( F \) such that \( x \notin F \) there exist a \( \tau_1 \)-open set \( U \) and a \( \tau_2 \)-open set \( V \) such that \( x \in U, F \subseteq V \) and \( U \cap V = \emptyset \). \( (X, \tau_1, \tau_2) \).

Definition 2-10. A bitopological space \( (X, \tau_1, \tau_2) \) is said to be \( \mathcal{O} \)-regular if for each point \( x \in X \) and for each i-open set \( V \) of \( X \) containing \( x \), there exists an \( \mathcal{O} \)-open set \( U \) such that \( x \in U \subseteq \mathcal{O} \)-cl \( (U) \subseteq V \). and is said to be \( \mathcal{O} \)-regular if for each point \( x \in X \) and for each \( \mathcal{O} \)-open set \( V \) of \( X \).

Definition 2-11. A space \( (X, \tau_1, \tau_2) \) is said to be pairwise Lindelof (p-Lindelof) if each pairwise open cover \( \mu \), (i.e. \( \mu \subseteq \tau_1 \cup \tau_2 \), \( \mu \cap \tau_1 \cap \tau_2 \neq \emptyset \)).

In a topological space we know that for every Lindelof and countably compact topological space is pairwise compact.

Proposition 2-12. Every \( \mathcal{O} \)-regular almost countably pairwise compact space is pairwise compact.

Proof: Let \( X \) be a \( \mathcal{O} \)-regular almost countably pairwise compact space, \( \mathcal{O} \)-Lindelof space and
1,2,...,m} since X is almost countably pairwise compact clearly (B_{X0k} : k = 1,2,...,m) is a finite subcover of A which completes the proof.

homeomorphism) if the induced functions \( f: (X, \tau_1) \rightarrow (Y, \tau_Y) \) are continuous (open, closed, homeomorphism), \[5\]

Since X is almost countably pairwise compact, there exists a finite subset \( \{ n_i : i = 1,2,3,...,m \} \) such that \( \bigcup \{ \text{cl}(f^n(U_i)) : i = 1,2,...,m \} = X \). Hence \( Y = f(X) = f(\bigcup \{ \text{cl}(f^n(U_i)) : i = 1,2,...,m \}) = \bigcup \{ f(\text{cl}(f^n(U_i))) : i = 1,2,...,m \} = \bigcup \{ \text{cl}(f(f^n(U_i))) : i = 1,2,...,m \} = \bigcup \{ \text{cl}(U_i) : i = 1,2,...,m \} \), This shows that Y is almost countably compact.

Proposition 2.14. A continuous image of an almost countably pairwise compact space is almost countably pairwise compact.

Proof: Suppose that \( X \) is an almost countably pairwise compact space and let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau_Y) \) be a continuous function. Let \( U = \{ U_n : n \in \omega \} \) be a countable pairwise open cover of Y. Then \( V = \{ f^{-1}(U_n) : U_n \in U \} \) is a countable pairwise open cover of X.

Definition 2.15. A map \( f \) from a space \( (X, \tau_1, \tau_2) \) to a space \( (Y, \tau_Y) \) is called almost i-open function if \( f^{-1}(\text{cl}(U)) \subseteq \text{cl}(f^{-1}(U)) \) for all \( U \in \tau_Y \).

Definition 2.16. Let \( f: X \rightarrow Y \) be a closed continuous surjective map such that \( f'(y) \) is compact for each \( y \in Y \), then \( f \) is called a perfect map, \[9\].

That \( f'(y) \) is compact for each \( y \in Y \), then \( f \) is called an i-perfect map. For every \( y \in Y \), there exists \( V_y \in \mathcal{V} \) such that \( f'(y) \subseteq V_y \). Since \( f'(y) \) is compact, then \( W_{n_i} = f(X_i \setminus V_{n_i}) \) is an open neighborhood of \( y \). Since \( Y \) is almost countably pairwise compact, then there exists a finite subfamily \( \{ w_{n_i} : i = 1,2,...,m \} \) of \( W \) such that \( Y = \bigcup \{ \text{cl}(w_{n_i}) : i = 1,2,...,m \} \). Since \( f \) is almost open, then \( X = f'(Y) = f'(\bigcup \{ w_{n_i} : i = 1,2,...,m \}) \subseteq \bigcup \{ f', \text{cl}(w_{n_i}), i = 1,2,...,m \} \subseteq \bigcup \{ f', \text{cl}(V_{n_i}), i = 1,2,...,m \} \) and since every element of \( V \) is the union of a finite subfamily of \( \mu \). This shows that \( X \) is almost countably pairwise compact, which completes the proof.

With the same sense we give the following definition,

Definition 2.17. Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau_Y, \tau_{Y'}) \) be an i-continuous surjective map such that \( f : X \rightarrow Y \) is almost countably pairwise compact space and \( f: X \rightarrow Y \) be an almost i-open and i-perfect mapping. Then \( X \) is almost countably pairwise compact.

Proof: Let \( \mu \) be a countable pairwise open cover of \( X \) and let \( \mathcal{V} = \{ V : \text{there exists a finite subfamily} J \text{ of} \mu \text{ such that} V = \bigcup J \} \) Then \( \mathcal{V} \) is countable, since \( \mu \) is countable. Hence we can enumerate \( \mathcal{V} \) as \( \{ V_n : n \in \omega \} \). For each \( n \in \omega \), let \( w_n = Y_n \setminus f(X_i \setminus V_n) \), then \( w_n \) is an i-open subset of \( Y_i \) for each \( i = 1,2,3,...,m \) since \( f \) is i-closed. Let \( \mathcal{W} = \{ w_n : n \in \omega \} \), then \( \mathcal{W} \) is a countable pairwise open cover of \( Y \). In fact,
Reference:

[2] - M. Bonanzinga, M.V. Matveev, C.M. Pareek, Some remarks on generalizations of countably compact spaces and Lindelof spaces,
[5] - Elsevier B.V. bitopological spaces: theory, relations with generalized algebraic structures and applications, British library

Due to the language barrier, the abstract appears to be a scientific paper in Arabic. The abstract discusses some aspects of topology, specifically focusing on the concept of semi uniformity and its relationship with other topological concepts. The author, Mr. Ali, a native of Iraq, explores these relationships and their implications in the field of mathematics.