Some properties of a new subclass of multivalent analytic functions with negative coefficients
Involving the generalized Noor integral operator

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Abstract. In this paper, we study a new subclass $A^{\lambda,q}_{\alpha,p,m}(\gamma,\mu,\eta,a,b,c)$ of multivalent analytic functions with negative coefficients defined in the unit disk by making use of the generalized Noor integral operator. We obtain some geometric properties for this class, like coefficient estimate, extreme points, inclusive property, radii of starlikeness and convexity, Hadamard product and weighted mean.

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1. Introduction

Let $A(p,m)$ denote the class of all functions of the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p},$$

where $p, m \in 1, 2, \ldots$ ...

which are analytic and multivalent in the open unit disk $U = \{z \in C : |z| < 1\}$.

Let $k(p,m)$ denote the subclass of $A(p,m)$ consisting of functions analytic and multivalent which can be expressed in the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p},$$

with $a_{n+p} \geq 0; p, m \in 1, 2, \ldots$ ...

For the functions $f \in k(p,m)$ given by (1, 2) and $g \in k(p,m)$ defined by:

$$g(z) = z^p + \sum_{n=m}^{\infty} b_{n+p} z^{n+p},$$

we define the Hadamard product (or convolution) $f^* g$ of $f$ and $g$ is defined (as usual) by:

$$(f^* g)(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g^* f)(z)$$

For real or complex numbers $a, b, c \in \{0, 1, -1, -2, \ldots\}$, the hypergeometric series is defined by:

$$2 F_1(a, b, c, z) = 1 + \frac{ab}{c} \frac{z}{1} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \ldots$$

$$= \frac{z^2}{2!} + \ldots$$

We note that the series in (3) converges absolutely for all $z \in U$ so that it represents and analytic function in $U$.

The authors [2] introduced a function

$$(x^p 2 F_1(a, b, c, z))^l$$

given by

$$= \left(\sum_{n=m}^{\infty} a_{n+p} z^{n+p}\right)^l$$

$$= \frac{x^p}{(1-z)^{x+p}} (l > -p),$$

which leads us to the following family of linear operators:

$$I_{p,m}^l(a, b, c) f(z) = \left(x^p 2 F_1(a, b, c, z)^l\right) f(z),$$

where
\[
I^{\lambda}_{p,k}(a, b, c; f(z)) = z^p - \sum_{n,m} \frac{(c)(\lambda+p)_a}{(a)_n(b)_n} a_{n+p} z^{p+n},
\]
where \((x)_k\) denote the pochhammer symbol defined by
\[
(x)_0 = \text{lnd}(x)_k = x(x+1)(x+2)...(x+k-1),
\]
\[k \in \mathbb{N}.
\]

It is easily verified from (6) that
\[
\alpha(I^{\lambda}_{p,k}(a, b, c; f(z))) = (\lambda + p)I^{\lambda+1}_{p,k}(a, b, c; f(z)) - \lambda I^{\lambda}_{p,k}(a, b, c; f(z))
\]
Differentiating above, \(q\) -times, we get
\[
\alpha(I^{\lambda}_{p,k}(a, b, c; f(z)))^{(q)} = (\lambda + p)I^{\lambda+1}_{p,k}(a, b, c; f(z))^{(q)} - (\lambda + q)(I^{\lambda}_{p,k}(a, b, c; f(z)))^{(q)}
\]
where \(q \in \mathbb{N}_0, q < p\) and for each \(f \in k(p, m)\) we have
\[
I^{(q)}(z) = S(p, q) z^{p-q} - \sum_{n,m} S(n+p, q) a_{n+p} z^{n+p},
\]
where
\[
S(p, q) = \frac{p!}{(p-q)!}.
\]
For two functions \(f\) and \(g\) analytic in \(U\), we say that the function \(f\) is subordinate to \(g\) in \(U\), and write \(f \prec g\) \((z \in U)\) if there exists a

Schwarz function \(w(z)\), which is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in U)\)
Such that
\[
f(z) = g(w(z)) \quad (z \in U).
\]
Indeed, it is known that
\[
f(z) < g(z) \quad (z \in U) \Rightarrow f(0) = g(0),
f(U) \subset g(U).
\]
Furthermore, if the function \(g\) is univalent in \(U\), then we have the following equivalence:
\[
f(z) < g(z) \quad (z \in U) \Leftrightarrow f(0) = g(0), \quad f(U) \subset g(U)
\]
By making use of the Noor integral operator \(I^{\lambda}_{p,k}(a, b, c)\) and the above mentioned principle of subordination between analytic functions, we introduce and investigate the following new subclass of the class \(k_p\) of \(p\) -valent analytic functions.

**Definition 1.** A function \(f \in k(p, m)\) is said to be in the class \(Ak^{\lambda,q}_{p,m}(\gamma, m, \eta, a, b, c)\) if it satisfies
\[
(\lambda + p) \left( \frac{I^{\lambda+1}_{p,m}(a, b, c; f(z))^{(q)}}{I^{\lambda}_{p,m}(a, b, c; f(z))^{(q)}} \right) - (\lambda + q) \left( \frac{I^{\lambda+1}_{p,m}(a, b, c; f(z))^{(q)}}{I^{\lambda}_{p,m}(a, b, c; f(z))^{(q)}} \right)
\]
where
\[
p, m \in \mathbb{N}, q \in \mathbb{N}_0, p > q, 0 \leq \eta < 1, 0 \leq m \leq 1,
\]
\[0 \leq \eta < 1, a, b, c \in \mathbb{R} \setminus \mathbb{Z}^+ \text{ and } \lambda > -p
\]
By the definition of differential subordination and (6), (7) are equivalent to the following condition:

\[
\frac{\alpha(I^{\lambda}_{p,m}(a, b, c; f(z)))^{(q+1)} - (p-q)(I^{\lambda+1}_{p,m}(a, b, c; f(z))^{(q)}}{\gamma z(I^{\lambda+1}_{p,m}(a, b, c; f(z))^{(q+1)} + (m-\eta)(I^{\lambda}_{p,m}(a, b, c; f(z))^{(q))}} < 1, \quad (z \in U)
\]

2. Coefficient Estimates

**Theorem 2.1** Let \(f \in k(p, m)\) be defined by
\[
(1.2).\text{ Then } f \in Ak^{\lambda,q}_{p,m}(\gamma, m, \eta, a, b, c)\text{ if and only if}
\]
\[ \sum_{m=0}^{n} \frac{\binom{n}{m} (n+1+\gamma+\gamma(p-q)(\lambda+p)\eta)_{m+n} a_{n+p}}{(n+p-q)!} \left( \frac{p^{m+\gamma}(p-q)}{(p-q)!} \right) \]

where

\[ p, q \in \mathbb{N}, \quad p \leq q, 0 \leq \gamma < 1, 0 < m \leq 1, \]

0 \leq \eta < 1, a, b, c \in R \text{, and } \lambda > -p \]

The result is sharp.

**Proof.** Assume that inequality (2.1) holds true and \( |z| = 1 \). Then, we obtain

\[ \left| \frac{\sum m(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \right| \]

\[ \leq \sum_{m=0}^{n} \frac{n(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \]

\[ \leq \sum_{m=0}^{n} \frac{n(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \]

by hypothesis. Hence, by maximum modulus principle, we have \( f \in A \hat{K}_{p,m}^{\gamma}(\gamma, m, \eta, a, b, c) \).

To show the converse, let \( f \in A \hat{K}_{p,m}^{\gamma}(\gamma, m, \eta, a, b, c) \). Then

\[ \left| \frac{\sum m(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \right| \]

\[ \leq \sum_{m=0}^{n} \frac{n(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \]

Since \( \Re(z) \leq |z| \) for all \( z \), we have

\[ \Re\left( \sum_{m=0}^{n} \frac{n(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \right) < 1. \quad (10) \]

Now choosing the value of \( z \) on the real axis so that

\[ \frac{\sum m(n+p)!\binom{n}{m} (\lambda+p)_{m+n} a_{n+p} z^{m+p-q}}{(n+p-q)!} \] is real. Upon clearing the denominator of (10) and letting \( z \to 1 \) through real values, we obtain the inequality (9). Finally, the result (9) is sharp for the function

\[ f(z) = z^{p-q} \frac{p(n+p-q)!((m-\eta+\gamma(p-q))(\lambda+p)_{n} b_{n})}{(p-q)!((n+1+\gamma+\gamma(p-q))(\lambda+p)_{n})} z^{m+p-q}, (n \geq m, m \in \mathbb{N}) \quad (11) \]

**3. Extreme points and inclusive property**

In this section, we obtain extreme points and inclusive property for the class...
Theorem 3. Let \( f_p(z) = z^p \) and

\[
f(z) = z^p - \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))} z^{p+1}, \quad (n \geq m, m \in N).
\]

Then \( f \in AK_{p,m}^{\lambda,q}(\gamma, m, \eta, a, b, c) \) if and only if can be expressed in the form

\[
f(z) = \Theta pz^p + \sum_{n=m}^{\infty} \Theta_{n+p} f_{n+p}(z), \quad \ldots \quad (12)
\]

where \( \Theta_p \geq 0, \Theta_{n+p} \geq 0 \) and \( \Theta_p + \sum_{n=m}^{\infty} \Theta_{n+p} = 1 \).

Proof. Suppose that \( f \) can be express as in (9).

\[
f(z) = \Theta pz^p + \sum_{n=m}^{\infty} \Theta_{n+p} z^p - \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))} z^{p+1},
\]

\[
z^p - \sum_{n=m}^{\infty} \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))} \Theta_{n+p} z^{p+1}.
\]

Now

\[
\sum_{n=m}^{\infty} \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))} \Theta_{n+p} = \sum_{n=m}^{\infty} \Theta_{n+p} = 1 - \Theta_p \leq 1.
\]

This shows that \( AK_{p,m}^{\lambda,q}(\gamma, m, \eta, a, b, c) \).

By (1), we have

Conversely, Assume that \( AK_{p,m}^{\lambda,q}(\gamma, m, \eta, a, b, c) \).

\[
a_{n+p} = \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))}, \quad (n \geq m).
\]

Therefore, we can set

\[
\Theta_{n+p} = \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))} a_{n+p}, (n \geq m)
\]

and \( \Theta_p = 1 - \sum_{n=m}^{\infty} \Theta_{n+p} \). Then

\[
f(z) = z^p - \sum_{n=m}^{\infty} a_{n+p} z^{n+p}
\]

\[
z^p - \sum_{n=m}^{\infty} \frac{pl((n + p - q)!((m - \eta + y(p - q))(a_n)(b_n))}{(p - q)!(n + p)!((n(1 + \gamma) + m - \eta + y(p - q))(c_\lambda)(\lambda + p))} \Theta_{n+p} z^{n+p} \quad \text{that is the required representation.}
\]

Theorem 4. Let \( 0 \leq \eta < 1, 0 < m \leq 1, 0 \leq \eta < 1 \), \( a, b, c \in \mathbb{R} \setminus \mathbb{Z} \) and \( \lambda > p \). Then

\[
AK_{p,m}^{\lambda,q}(\gamma, m, \eta, a, b, c) \subset AK_{p,m}^{\lambda,q}(\gamma, m, \eta, a, b, c),
\]

where

\[
\sigma = \frac{(\lambda + p + 1)(1 + m - \eta + y(p - q + 1)) - (\lambda + p)(1 + m + y(p - q + 1))(m - \eta + y(p - q))}{(\lambda + p + 1)(1 + m - \eta + y(p - q + 1)) - (\lambda + p)(m - \eta + y(p - q))}, \quad \ldots \quad (13)
\]
Proof. Let the function \( f \) given by (2) belong to the class \( AK^{q}_{p,m}(\gamma, m, \eta, a, b, c) \). Then, by
\[
\sum_{n=m}^{\infty} \frac{(p-q)!((n+p)!(n+1+\gamma+m-\eta+\gamma(p-q))(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b)_n} a_{n+p} \leq 1
\]
In order to prove that \( AK^{q}_{p,m}(\gamma, m, \sigma, a, b, c) \), we must have
\[
\sum_{n=m}^{\infty} \frac{(p-q)!((n+p)!(n+1+\gamma+m-\eta+\gamma(p-q))(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b)_n} a_{n+p} \leq 1
\]
Note that (14) is satisfies if
\[
\frac{(p-q)!((n+p)!(n+1+\gamma+m-\eta+\gamma(p-q))(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b)_n} a_{n+p} \leq \frac{(p-q)!((n+p)!(n+1+\gamma+m-\eta+\gamma(p-q))(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b)_n} a_{n+p}.
\]
Rewriting the inequality (15), we have
\[
\sigma = \frac{(\lambda+p+1)(1+m-\eta+\gamma(p-q+1))-(\lambda+p)(1+m-\eta+\gamma(p-q+1))(m-\eta+\gamma(p-q))}{(\lambda+p+1)(1+m-\eta+\gamma(p-q+1))-(\lambda+p)(m-\eta+\gamma(p-q))}
\]
\[\text{(n \geq m), m \in N}\]
Since the right-hand side of (16) is an increasing function of \( n \), thus we get (13) and this completes the proof.

4. Radii of starlikeness and convexity

Theorem 4. If \( AK^{q}_{p,m}(\gamma, m, \sigma, a, b, c) \), then \( f \) is
\[
r_{1} = \inf_{n} \left\{ \frac{(p-\rho)!((n+p)!(n+1+\gamma+m-\eta+\gamma(p-q))(c)_n(\lambda+p)_n}{(n+p-q)!p((n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b)_n} a_{n+p} \right\}^{\frac{1}{n}} (n \geq m).
\]
The result is sharp for the function \( f \) given by (11).

Proof. It is sufficient to show that
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \rho \forall \rho < r_{1}, \quad (17)
\]
But
\[
\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{-\sum_{n=m}^{\infty} na_{n+p}z^{n+p}}{z^{p} - \sum_{n=m}^{\infty} na_{n+p}z^{n+p}} \right| \leq \left| \sum_{n=m}^{\infty} na_{n+p}|z^{p-n}| \right| \frac{\sum_{n=m}^{\infty} na_{n+p}|z^{p}|}{1 - \sum_{n=m}^{\infty} na_{n+p}|z^{p}|}.
\]
Thus (17) will be satisfied if

\[
\sum_{n=m}^{\infty} na_{n+p}|z^{p}| \leq \left| z^{p} - \sum_{n=m}^{\infty} na_{n+p}z^{n+p} \right| \leq \sum_{n=m}^{\infty} na_{n+p}|z^{p}| \leq \left| z^{p} - \sum_{n=m}^{\infty} na_{n+p}z^{n+p} \right|
\]
with the said of (9), (18) is true if

\[
\sum_{n=m}^{\infty} \frac{(n+p-\rho)}{(p-\rho)} a_{n+p}|z^{n}| \leq 1,
\]
and if

\[
\sum_{n=m}^{\infty} \frac{(n+p-\rho)}{(p-\rho)} a_{n+p}|z^{n}| \leq 1.
\]
which follows the result.

**Theorem 5.** If $AK_{p,m}^{k,q}(\gamma, m, \sigma, a, b, c)$, then $f$ is convex of order $\rho(0 \leq \rho < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_{z} \left\{ \frac{p(p-p)(p-q)!((n+1)! + m-\eta + \gamma(p-q))(c)_n(\lambda + p)_n}{(n+p)(n+p-\rho)!p(n+p-q)!((m-\eta + \gamma(p-q))(a)_n(b)_n)} \right\}^{1/n}, (n \geq m).$$

The result is sharp for the function $f$ given by (11).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f'(z)} + 1 - p \right| \leq p - \rho \forall |z| < r_2. \quad \ldots \quad (19)$$

But

$$\left| \frac{zf'(z)}{f'(z)} + 1 - p \right| \leq p - \rho \forall |z| < r_2.$$

Thus (19) will be satisfied if

$$\sum_{n=m}^{\infty} \frac{\sum_{n=m}^{\infty} n(n+p) a_{n+p} |z|^n}{p - \sum_{n=m}^{\infty} (n+p) a_{n+p} |z|^n} \leq p - \rho,$$

or if

$$\sum_{n=m}^{\infty} \frac{(n+p) a_{n+p}}{p(p-\rho)} |z|^n \leq 1, \quad \ldots \quad (20)$$

with the aid of (9), (20) is true if

$$\left| z_{n+p} \right| \leq \frac{1}{2} [(1-j)f(z) + (1+j)g(z)], 0 < j < 1.$$

**5. Weighted mean**

**Definition 2.** Let $f$ and $g$ be in the class $AK_{p,m}^{k,q}(\gamma, m, \sigma, a, b, c)$. Then the Weighted mean $h_j$ of $f$ and $g$ is given by

$$h_j(z) = \frac{1}{2} [(1-j)f(z) + (1+j)g(z)], 0 < j < 1.$$

**Theorem 6.** Let $f$ and $g$ be in the class $AK_{p,m}^{k,q}(\gamma, m, \sigma, a, b, c)$. Then the Weighted mean $h_j$ of $f$ and $g$ is also in the class $AK_{p,m}^{k,q}(\gamma, m, \sigma, a, b, c)$

Proof. By Definition 2, we have

$$h_j(z) = \frac{1}{2} [(1-j)f(z) + (1+j)g(z)], 0 < j < 1.$$

$$= \frac{1}{2} [(1-j)(z^p - \sum_{n=m}^{\infty} a_{n+p} z^{n+p}) +$$

$$+ (1+j)(z^p - \sum_{n=m}^{\infty} b_{n+p} z^{n+p})], 0 < j < 1.$$

$$= z^p - \sum_{n=m}^{\infty} \frac{1}{2} ((1-j)a_{n+p} + (1+j)b_{n+p}) z^{n+p}$$

since $f$ and $g$ are in the class $AK_{p,m}^{k,q}(\gamma, m, \sigma, a, b, c)$, then by Theorem (2), we get

$$\sum_{n=m}^{\infty} (n+p)!((n+\gamma) + m-\eta + \gamma(p-q))(c)_n(\lambda + p)_n$$

$$\times a_{n+p} \leq p^m \frac{(p-\eta + \gamma(p-q))}{(p-q)!},$$

and
\[
\sum_{n=m}^{\infty} \frac{(n+\gamma)(n+\lambda + 1) + m - \eta + \gamma(p-q)(c_n)(\lambda + p)_n}{(n + p - q)!} \times b_{n+p} \leq \frac{p!(m-\eta + \gamma(p-q))}{(p-q)!}.
\]

Hence
\[
\sum_{n=m}^{\infty} \frac{(n+\gamma)(n+\lambda + 1) + m - \eta + \gamma(p-q)(c_n)(\lambda + p)_n}{(n + p - q)!} \times a_{n+p} \leq \frac{1}{2} \frac{p!(m-\eta + \gamma(p-q))}{(p-q)!} \times \frac{1}{2} \frac{p!(m-\eta + \gamma(p-q))}{(p-q)!}.
\]

This shows that \( h_j \in \mathbb{A}^{k,q}_{\eta,m}(\gamma,m,\sigma,a,b,c) \)

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بعض الخصائص الجديدة للدوال التحليلية متعددة الكافؤ مع معاملات سالبة مرتبطة مع تعليم مؤثر Noor التكامل

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الخلاصة:
في هذا البحث، دراسة فئة فرعية جديدة من الدوال التحليلية متعددة الكافؤ مع معاملات السلبية المحددة في القرص الوحدة من خلال الاستفادة من مؤثر Noor Noor القطر Nstarlikeness Q ونصف قطر التحد وضرب هadamard. nاک

الكلمات الإفتتاحية:
دوال متعددة الكافؤ، النقاط الحرجة، انصاص الاقطار، ضرب Hadamard