Injective Modules Relative To a Preradical

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Abstract

The concept of $\rho$-injective modules (where $\rho$ is a preradical) is introduced in this work as a generalization of injective modules. The definition of $\rho$-injectivity unifies several definitions on generalizations of injectivity such as nearly injective modules and special injective modules. Many characterizations and properties of $\rho$-injectivity are given. We study the endomorphisms rings of $\rho$-injective modules. The results of this work unify and extend many results in the literature.

Keywords: Injective modules; nearly-injective modules; preradical; endomorphisms ring.

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1. Introduction:
Throughout this work, $R$ stands a commutative ring with identity element $1$ and a module means a unitary left $R$-modules. The class of all $R$-module will be denoted by $R$-Mod and the symbol $p$ means a preradical on $R$-Mod (A preradical $p$ is defined to be a subfunctor of the identity functor of $R$-Mod). For an $R$-module $M$, the notations $J(M)$, $L(M)$, $E(M)$ and $S = \text{End}_R(M)$ will respectively stand for the Jacobson radical of $M$, the prime radical of $M$, the injective envelope of $M$ and the endomorphism ring of $M$. The notation $\text{Hom}_B(N, M)$ denoted to the set of all $R$-homomorphism from $R$-module $N$ into $R$-module $M$. An $R$-module $M$ is called injective, if for every $R$-monomorphism $f : A \rightarrow B$ (where $A$ and $B$ are $R$-modules) and every $R$-monomorphism $g : A \rightarrow M$, there exists an $R$-homomorphism $h : B \rightarrow M$ such that $g = h \circ f$ [1].

Injective modules have been studied extensively, and several generalizations for these modules are given, for example, quasi-injective modules [2], P-injective Modules [3], and S-injective module [4].

In 2000, nearly-injective modules were discussed in [5] as generalization of injective modules. An $R$-module $M$ is said to be nearly injective if for each $R$-monomorphism $f : A \rightarrow B$ (where $A$ and $B$ are two $R$-modules), each $R$-homomorphism $g : A \rightarrow M$, there exists an $R$-homomorphism $h : B \rightarrow M$ such that $(h \circ f)(a) - g(a) \in \{1\}(M)$, for all $a \in A$ [5].

Also, in [6] M. S. Abbas and Sh. N. Abd-Alridha introduced the concept of special injective modules as a generalization of injectivity. An $R$-module $M$ is said to be special injective if for each $R$-monomorphism $f : A \rightarrow B$ (where $A$ and $B$ are two $R$-modules), each $R$-homomorphism $g : A \rightarrow M$, there exists an $R$-homomorphism $h : B \rightarrow M$ such that $(h \circ f)(a) - g(a) \in L(M)$, for all $a \in A$ [6]. A ring $R$ is called Von Neumann regular (in short, regular) if for each $a \in R$, there exists $b \in R$ such that $a = aba$. For a submodule $N$ of an $R$-module $M$ and $a \in M$, $[N:R]a = \{r \in R \mid ra \in N\}$. For an $R$-module $M$ and $a \in M$. A submodule $N$ of an $R$-module $M$ is called essential and denoted by $N \leq E M$ if every non zero submodule of $M$ has nonzero intersection with $N$.

2. Injective Modules Relative to a Preradical

In this section, we will introduce a new generalization of injective module namely, injective module relative to a preradical. We will study some properties and characterizations of these modules.

We start by the following definition:

**Definition 2.1.** Let $\rho$ be a preradical on $R$-Mod and let $M, N$ and $K$ be $R$-modules. A module $M$ is said to be $N$-injective relative to the preradical $\rho$ (shortly, $\rho$-$N$-injective) if for each $R$-monomorphism $f : K \rightarrow N$ and each $R$-homomorphism $g : K \rightarrow M$ there is an $R$-homomorphism $h : N \rightarrow M$ such that $(h \circ f)(x) - g(x) \in \rho(M)$, for each $x \in K$.

![Diagram](image)

An $R$-module $M$ is said to be injective relative to the preradical $\rho$ (shortly, $\rho$-injective) if $M$ is $\rho$-$N$-injective for all $R$-modules $N$. A ring $R$ is said to be $\rho$-injective ring, if $R$ is a $\rho$-injective $R$-module.

**Examples and Remarks 2.2.**

(1) It is clear that injective modules and $N$-injective modules are $\rho$-$N$-injective for every $R$-module $N$.

(2) There are many types of preradical functors, for examples: the Jacobson radical functor (J), the socle functor (soc), the prime radical functor (L) and the torsion functor (T) [7]. Each one of these functors gives a special case of $\rho$-injective modules, for example a left $R$-module $M$ is said...
to be (soc)-injunctive if $M$ is $\rho$-injective, where $\rho = \text{soc}$.

(3) The concept of nearly-injective module (which is introduced in [5]) is a special case of $\rho$-injective $R$-modules by taking $\rho = J$, where $J$ is the Jacobson radical functor.

(4) Special injective modules (which are introduced in [6]) are special case of $\rho$-injectivity by taking $\rho = L$, where $L$ is the prime radical functor.

(5) Let $M$ be a module such that $\rho(M) = 0$, thus $M$ is injective if and only if $M$ is $\rho$-injective.

(6) It is clear that if $\rho(M) = M$, then $M$ is a $\rho$-injective module, in particular:

(a) Every module $M$ which has no maximal submodule (i.e., $\text{J}(M) = M$) is $J$-injective.

(b) Every semisimple module $M$ (i.e., $\text{soc}(M) = M$) is (soc)-injective. Thus $\rho$-injective modules may not be injective, for example: let $M = \mathbb{Z}_p$ as $\mathbb{Z}$-module, where $p$ is a prime number. Since $M$ is semisimple, thus $\text{soc}(M) = M$ and hence $M$ is (soc)-injective but $M$ is not injective.

(7) Let $M_2$ be an $R$-module. If $M_2$ is a $\rho$-$N$-injective $R$-module and $M_1$ is isomorphic to $M_2$, then $M_2$ is a $\rho$-$N$-injective.

(8) Form (7) above we have that $\rho$-injectivity is an algebraic property.

(9) Every submodule of semisimple $R$-module is $\rho$-injective, where $\rho$ is the socle functor.

**Lemma 2.3.** Let $N$ and $M$ be $R$-modules. Then the following statements are equivalent:

1. $M$ is $\rho$-$N$-injective;
2. for any diagram,

$$
\begin{array}{ccc}
0 & \longrightarrow & A & \longrightarrow & N \\
& & g & \downarrow & \\
& & M & \longrightarrow & \end{array}
$$

where $A$ is a submodule of an $R$-module $N$, $g: A \rightarrow M$ is any $R$-homomorphism and $i$ is the inclusion mapping, there exists an $R$-homomorphism $h: N \rightarrow M$ such that $(h \circ i)(a) - g(a) \in \rho(M)$, for all $a$ in $A$.

**Proof:** The proof is obvious. $\square$

In the following proposition we show that the set of all essential submodules of $N$ is a test set for $\rho$-$N$-injectivity.

**Proposition 2.4.** Let $N$ be an $R$-module. Then an $R$-module $M$ is $\rho$-$N$-injective if and only if for each essential submodule $A$ of $N$ and each $R$-homomorphism $f: A \rightarrow M$, there is an $R$-homomorphism $g: N \rightarrow M$ such that $(g \circ i)(a) - f(a) \in \rho(M)$ for each $a$ in $A$.

**Proof:**

$(\Rightarrow)$ Let $A$ be any essential submodule of $N$ and $f: A \rightarrow M$ be any $R$-homomorphism.

Consider the diagram (1).

$$
\begin{array}{ccc}
0 & \longrightarrow & A & \longrightarrow & N \\
& & f & \downarrow & \\
& & M & \longrightarrow & \end{array}
$$

Let $A^c$ be any complement submodule of $A$ in $N$. By [8, p.16], we have that $A \oplus A^c \leq N$.

Define $g: A \oplus A^c \rightarrow M$ by $g(a + a_1) = f(a)$, for all $a \in A$ and $a_1 \in A^c$. It is easy to prove that $g$ is a well-defined $R$-homomorphism.

Therefore, we have the diagram (2).

$$
\begin{array}{ccc}
0 & \longrightarrow & A \oplus A^c & \longrightarrow & N \\
& & g & \downarrow & \\
& & M & \longrightarrow & \end{array}
$$

By hypothesis, there exists an $R$-homomorphism $h: N \rightarrow M$ such that $(h \circ i)(a) - g(x) \in \rho(M)$ for all $x$ in $A \oplus A^c$.

For the diagram (1), we get that $(h \circ i)(a) - f(a) = (h \circ i)(a) - g(a) \in \rho(M)$ for all $a$ in $A$. Therefore, $M$ is a $\rho$-$N$-injective $R$-module, by Lemma 2.3. $\square$

Now, we will study the direct product and the direct sum of $\rho$-$N$-injective modules.

**Proposition 2.5.** Let $(M_a)_{a \in A}$ be a family of $R$-modules. Then:

1. if $\prod_{a \in A} M_a$ is a $\rho$-$N$-injective (where $N$ is an $R$-module), then each $M_a$ is $\rho$-$N$-injective.
(2) If $\rho(\prod_{a \in A} M_a) = \prod_{a \in A}(\rho(M_a))$, then the converse of (1) is true.

Proof: (1) Put $M = \prod_{a \in A} M_a$ and let $i: M \to A$ and $p_a: M \to M_a$ be the injections and projections associated with this direct product respectively. Suppose that $M$ is $\rho$-N-injective. To prove that $M_a$ is $\rho$-N-injective for each $\lambda \in A$. Consider the following diagram where $A$ is a submodule of $N$ and $\alpha_\lambda$ is a $R$-homomorphism.

Since $M$ is a $\rho$-N-injective module, there exists an $R$-homomorphism $h: N \to M$ such that $(h \circ i)(a) = (i \circ \alpha_\lambda)(a) \in \rho(M)$ for all $a$ in $A$. Put $g_a = p_a \circ h: N \to M_a$. For every $a$ in $A$, we have that

$$\begin{align*}
(g_\lambda \circ i)(a) - \alpha_\lambda(a) &= g_\lambda(a) - \alpha_\lambda(a) = (p_\lambda \circ h)(a) - \alpha_\lambda(a) = (p_\lambda \circ i)(a) - (i \circ \alpha_\lambda)(a) = p_\lambda(h(a)) - (i \circ \alpha_\lambda)(a) \in \rho(M_a).
\end{align*}$$

Thus $(g_\lambda \circ i)(a) - \alpha_\lambda(a) \in \rho(M_a)$, for each $\lambda \in A$ and for every $a \in A$ and hence $M_a$ is $\rho$-N-injective, for each $\lambda \in A$.

(2) Suppose that $\rho(\prod_{a \in A} M_a) = \prod_{a \in A}(\rho(M_a))$ and consider the following diagram.

For each $\lambda \in A$, let $p_\lambda: M \to M_a$ be the projection $R$-homomorphism. Since each $M_a$ is $\rho$-N-injective, thus there exists an $R$-homomorphism $g_\lambda: N \to M_a$ for each $\lambda \in A$ such that $(g_\lambda \circ i)(a) - (p_\lambda \circ \alpha')(a) \in \rho(M_a)$, for every $a$ in $A$. Define $g: N \to M$ by $g(x) = \{g_\lambda(x)\}_{a \in A}$, for every $x \in N$. It is clear that $g$ is an $R$-homomorphism. For every $a$ in $A$, we have that

$$\begin{align*}
(g \circ i)(a) - \alpha(a) &= \{g_\lambda(i(a))\}_{a \in A} - \{(p_\lambda \circ \alpha')(a)\}_{a \in A} = \{(g_\lambda \circ i)(a) - (p_\lambda \circ \alpha')(a)\}_{a \in A} \in \prod_{a \in A}(\rho(M_a)).
\end{align*}$$

Since $\prod_{a \in A}(\rho(M_a)) = \rho(\prod_{a \in A} M_a)$ (by hypothesis) it follows that $(g \circ i)(a) - \alpha(a) \in \rho(M)$, for every $a$ in $A$. Therefore, $M$ is a $\rho$-N-injective module.

Corollary 2.6. Let $R$ be a ring such that $R/J(R)$ is a semisimple $R$-module, let $\{M_\lambda\}_{a \in A}$ be a family of $R$-modules and let $N$ be any $R$-module. Then $\prod_{a \in A} M_\lambda$ is $(\text{soc})$-N-injective if and only if $M_\lambda$ is $(\text{soc})$-N-injective, for each $\lambda \in A$.

Proof: Since $R/J(R)$ is a semisimple $R$-module, $\text{soc}(\prod_{a \in A} M_\lambda) = \prod_{a \in A} \text{soc}(M_\lambda)$ [7, Exercise (11), p.239]. Therefore, the result follows from Proposition 2.5.

Corollary 2.7. Let $R$ be a ring and let $I$ be a finitely generated ideal of $R$. Let $\{M_\lambda\}_{a \in A}$ be a family of $R$-modules and let $N$ be an $R$-module. Then $\prod_{a \in A} M_\lambda$ is $\rho_I$-N-injective if and only if $M_\lambda$ is $\rho_I$-N-injective.

Proof: Since $I$ is a finitely generated ideal of $R$ it follows from [9, Exercise 3(i), p.174] that

$$\begin{align*}
I(\prod_{a \in A} M_\lambda) &= \prod_{a \in A}(I(M_\lambda))
\end{align*}$$

and hence

$$\begin{align*}
\rho_I(\prod_{a \in A} M_\lambda) &= \prod_{a \in A}(\rho_I(M_\lambda)).
\end{align*}$$

Therefore, the result follows from Proposition 2.5.

For any family $\{M_\lambda\}_{a \in A}$ of $R$-modules, if $\bigoplus_{a \in A} M_\lambda$ is an $N$-injective $R$-module, then each $M_\lambda$ is an $N$-injective and the converse is true, if $A$ is finite by [3, Proposition(1.11), p. 6].

The following proposition shows that this result is true in case of $\rho$-N-injectivity.

Proposition 2.8. Let $\{M_\lambda\}_{a \in A}$ be a family of $R$-modules, let $M = \bigoplus_{a \in A} M_\lambda$ and let $N$ be any $R$-module.
(1) If $M$ is $\rho$-$N$-injective, then each $M_\lambda$ is $\rho$-$N$-injective.

(2) If $\Lambda$ is a finite set, then the converse of (1) is true.

**Proof:** Suppose that $M$ is a $\rho$-$N$-injective module. To prove that each $M_\lambda$ is $\rho$-$N$-injective.

(1) Let $i_2: M_2 \rightarrow M$ and $p_2: M \rightarrow M_2$ be the injections and projections associated with this direct product respectively. Consider the following diagram, where $A$ is a submodule of $N$ and $\alpha_2$ is an $R$-homomorphism.

Since $M$ is $\rho$-$N$-injective, there exists an $R$-homomorphism $h: N \rightarrow M$ such that $(h \circ i_2)(a) = (i_2 \circ \alpha_2)(a) \in \rho(M)$, for all $a$ in $A$. For each $\lambda \in \Lambda$, put $g_\lambda = p_\lambda \circ h: N \rightarrow M_\lambda$.

For every $a$ in $A$, we have that $(g_\lambda \circ i)(a) - \alpha_2(a) = g_\lambda(a) - \alpha_2(a) = (p_\lambda \circ h)(a) - \alpha_2(a) = (p_\lambda \circ h)(a) - (p_\lambda \circ i_2 \circ \alpha_2)(a) = (p_\lambda \circ h)(a) - (p_\lambda \circ i_2)(\alpha_2(a)) = p_\lambda(h(a)) - (i_2 \circ \alpha_2)(a) \in \rho(M_\lambda)$ (because $\rho$ is a preradical). Thus $g_\lambda(a) - \alpha_2(a) \in \rho(M_\lambda)$, for each $\lambda \in \Lambda$ and for every $a \in A$.

Therefore, $M_\lambda$ is $\rho$-$N$-injective, for each $\lambda \in \Lambda$.

(2) Suppose that $\Lambda$ is a finite set. Let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of $\rho$-$N$-injective modules. Since $\Lambda$ is finite it follows from [7, p.82] that

$\bigoplus_{\lambda \in \Lambda} M_\lambda = \prod_{\lambda \in \Lambda} M_\lambda$.

Since $\rho(\bigoplus_{\lambda \in \Lambda} M_\lambda) = \bigoplus_{\lambda \in \Lambda} \rho(M_\lambda)$ (by [10, Proposition 2, p.76]) it follows that $\rho(\prod_{\lambda \in \Lambda} M_\lambda) = \prod_{\lambda \in \Lambda} \rho(M_\lambda)$. By Proposition 2.5 (2), $\prod_{\lambda \in \Lambda} M_\lambda$ is $\rho$-$N$-injective and hence $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is $\rho$-$N$-injective. □

The following corollary is immediate from Proposition 2.8(1).

**Corollary 2.9.** Let $M$ be a $\rho$-$N$-injective $R$-module and let $K$ be a direct summand of $M$. Then $K$ is a $\rho$-$N$-injective $R$-module. □

**Corollary 2.10.** Let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of $R$-modules and let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$.

(i) If $\rho$ is a preradical and $M/\rho(M)$ is $\rho$-$N$-injective, then each $M_\lambda/\rho(M_\lambda)$ is $\rho$-$N$-injective.

(ii) If $\rho$ is a radical and $M/\rho(M)$ is $\rho$-$N$-injective, then each $M_\lambda/\rho(M_\lambda)$ is $\rho$-$N$-injective.

Proof: (i) Suppose that $\rho$ is a preradical and $M/\rho(M)$ is a $\rho$-$N$-injective $R$-module.

Since $M/\rho(M) = \bigoplus_{\lambda \in \Lambda} (M_\lambda/\rho(M_\lambda))$ and $M/\rho(M)$ is $\rho$-$N$-injective (by hypothesis) it follows that $\bigoplus_{\lambda \in \Lambda} (M_\lambda/\rho(M_\lambda))$ is $\rho$-$N$-injective. By Proposition 2.8(1), $M_\lambda/\rho(M_\lambda)$ is $\rho$-$N$-injective, for all $\lambda \in \Lambda$.

(ii) Suppose that $\rho$ is a radical and $M/\rho(M)$ is a $\rho$-$N$-injective $R$-module. By (i), $M_\lambda/\rho(M_\lambda)$ is $\rho$-$N$-injective, for all $\lambda \in \Lambda$.

Since $\rho$ is a radical, $\rho(M_\lambda/\rho(M_\lambda)) = 0$ and hence $M_\lambda/\rho(M_\lambda)$ is $N$-injective, for all $\lambda \in \Lambda$.

Proof: (ii) Suppose that $\rho$ is a preradical, each $M_\lambda/\rho(M_\lambda)$ is $\rho$-$N$-injective and $\Lambda$ is a finite set.

By Proposition 2.8(2), $\bigoplus_{\lambda \in \Lambda} (M_\lambda/\rho(M_\lambda))$ is $\rho$-$N$-injective. Since $\bigoplus_{\lambda \in \Lambda} (M_\lambda/\rho(M_\lambda)) = \bigoplus_{\lambda \in \Lambda} M_\lambda/\bigoplus_{\lambda \in \Lambda} \rho(M_\lambda) = M/\rho(M)$ it follows that $M/\rho(M)$ is $\rho$-$N$-injective.

Proof (ii) Suppose that $\rho$ is a radical, each $M_\lambda/\rho(M_\lambda)$ is $\rho$-$N$-injective and $\Lambda$ is a finite set.

By (ii), $M/\rho(M)$ is $\rho$-$N$-injective. Since $\rho$ is a radical, $\rho(M_\lambda/\rho(M_\lambda)) = 0$ and hence $M_\lambda/\rho(M_\lambda)$ is $N$-injective. □
Examples 2.11.

(1) The converse of Proposition 2.8(1) is not true in general. For example, let \( A \) be an infinite countable index set and set \( T_\lambda = Q \) for all \( \lambda \in A \) (where \( Q \) is the field of rational numbers).

Let \( R = \prod_{\lambda \in A} T_\lambda \) be the ring product of the family \( \{ T_\lambda \mid \lambda \in A \} \). It is easy to prove that \( R \) is a regular ring. For \( k \in A \), let \( e_k \) be the element of \( R \) whose kth-component is 1 and whose remaining components are 0.

Let \( A = \oplus_{\lambda \in A} R e_\lambda \), it is clear that \( A \) is a submodule of an \( R \)-module \( R \). By [7, p.140], \( A \) is a direct sum of injective \( R \)-modules, but \( A \) is not injective \( R \)-module. Since every injective \( R \)-module is \( \rho \)-injective, thus \( A \) is a direct sum of \( \rho \)-injective \( R \)-modules. Let \( \rho \) be any \( J \)-preradical. Assume that \( A \) is \( \rho \)-injective. Since \( R \) is a regular ring, thus \( J(A) = 0 \) (by [7, p.272]). Since \( \rho \) is a \( J \)-preradical, thus \( \rho(A) = 0 \) and hence \( A \) is injective and this is a contradiction. Thus \( A \) is not \( \rho \)-injective.

Therefore, \( A \) is a direct sum of \( \rho \)-injective modules, but it is not \( \rho \)-injective.

(2) Let \( M = Q \oplus Z \). Thus \( M \) is not \( \rho \)-injective \( Z \)-module, where \( \rho \) is a \( J \)-preradical. In fact, if \( M \) is \( \rho \)-injective, then by Proposition 2.8(1) we have \( Z \) is \( \rho \)-injective \( Z \)-module and hence \( Z \) is an injective \( Z \)-module (because \( \rho(Z) = J(Z) = 0 \) and this is a contradiction. Thus \( M \) is not \( \rho \)-injective \( Z \)-module.

In following, we will introduce further characterizations of \( \rho \)-injective modules.

Recall that a submodule \( N \) of an \( R \)-module \( M \) is said to be a direct summand of \( M \) if there exists a submodule \( K \) of \( M \) such that \( M = N \oplus K \), (i.e., \( M = N + K \) and \( N \cap K = 0 \)) [7]. This is equivalent to saying that, for every commutative diagram with exact rows,

\[
0 \longrightarrow N \overset{\beta}{\longrightarrow} M
\]

(where \( A \) and \( B \) are two \( R \)-modules), there exists an \( R \)-homomorphism \( h: B \to N \) such that \( f = h \circ \alpha \) [11]. It is well-known that an \( R \)-module \( M \) is injective if and only if \( M \) is a direct summand of every extension of itself [1, Theorem (2.1.5)].

For analogous result for \( \rho \)-injective \( R \)-modules, we introduce the following concept as a generalization of direct summands.

**Definition 2.12.** A submodule \( N \) of an \( R \)-module \( M \) is said to be \( \rho \)-direct summand of \( M \) if for every commutative diagram with exact rows,

\[
0 \longrightarrow N \overset{\beta}{\longrightarrow} M
\]

(where \( A \) and \( B \) are two \( R \)-modules), there exists an \( R \)-homomorphism \( h: B \to N \) such that \( (h \circ \alpha)(\alpha) - f(\alpha)e \rho(N) \), for all \( \alpha \in A \).

**Proposition 2.13.** Let \( N \) be a submodule of an \( R \)-module \( M \). Then the following statements are equivalent:

1. \( N \) is \( \rho \)-direct summand of \( M \);
2. for each diagram with exact row,

\[
0 \longrightarrow N \overset{\alpha}{\longrightarrow} M
\]

where \( I_N \) is the identity homomorphism of \( N \), there exists an \( R \)-homomorphism \( h: M \to N \) such that \( (h \circ \alpha)(\alpha) - a \in \rho(N) \), for all \( \alpha \in A \).

**Proof:** (1) \( \Rightarrow \) (2) Suppose that \( N \) is a \( \rho \)-direct summand of \( M \) and consider the following diagram with exact row:

\[
0 \longrightarrow N \overset{\alpha}{\longrightarrow} M
\]

where \( I_N \) is the identity homomorphism of \( N \), there exists an \( R \)-homomorphism \( h: M \to N \) such that \( (h \circ \alpha)(\alpha) - a \in \rho(N) \), for all \( \alpha \in A \).
Thus we have the following commutative diagram with exact rows.

\[ \begin{array}{ccc}
0 & \longrightarrow & N \\
\downarrow l_N & & \downarrow h \\
0 & \longrightarrow & M
\end{array} \]

By hypothesis, there exists a homomorphism \( h: M \to N \) such that \( (h \circ \alpha)(a) - l_N(a) \in \rho(N) \), for all \( a \) in \( A \) and hence \( (h \circ \alpha)(a) - a \in \rho(N) \), for all \( a \) in \( N \).

(2) \implies (1) Consider the following commutative diagram with exact rows.

\[ \begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f & & \downarrow g \\
0 & \longrightarrow & B \\
\downarrow \beta & & \downarrow h \\
0 & \longrightarrow & M
\end{array} \]

Thus we have the following diagram.

\[ \begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f & & \downarrow h \\
0 & \longrightarrow & B \\
\downarrow \beta & & \downarrow g \\
0 & \longrightarrow & M
\end{array} \]

By hypothesis, there exists a homomorphism \( h: M \to N \) such that \( (h \circ \beta)(a) - a \in \rho(N) \), for all \( a \in N \). Put \( h_1 = h \circ g: B \to N \). It is clear that \( h_1 \) is a homomorphism. Let \( a \in A \), thus \( (h_1 \circ \alpha)(a) - f(a) = (h \circ g) \circ \alpha)(a) - f(a) = (h \circ (g \circ \alpha))(a) - f(a) = (h \circ \beta)(f(a)) - f(a) = (h \circ \beta)(f(a)) - f(a) = \rho(N) \), for all \( a \) in \( A \) and this implies that \( N \) is a \( \rho \)-direct summand of \( M \). \( \square \)

In the following theorem we will give a characterization of \( \rho \)-injective modules, by using \( \rho \)-direct summands.

**Theorem 2.14.** For an \( R \)-module \( M \), the following statements are equivalent:

1. \( M \) is a \( \rho \)-injective.
2. \( M \) is a \( \rho \)-direct summand of every extension of itself.
3. \( M \) is a \( \rho \)-direct summand of every injective extension of itself.
4. \( M \) is a \( \rho \)-direct summand of at least, one injective extension of itself.
5. \( M \) is a \( \rho \)-direct summand of \( E(M) \), where \( E(M) \) is the injective hull of \( M \).

Proof: (1) \implies (2) Suppose that \( M \) is a \( \rho \)-injective \( R \)-module and let \( M_1 \) be any extension \( R \)-module of \( M \). We will prove that \( M \) is a \( \rho \)-direct summand of \( M_1 \). Consider the following diagram with exact row.

\[ \begin{array}{ccc}
0 & \longrightarrow & M \\
\downarrow l_M & & \downarrow f \\
0 & \longrightarrow & M_1 \\
\downarrow g & & \downarrow h \\
0 & \longrightarrow & M
\end{array} \]

Since \( M \) is \( \rho \)-injective, there exists an \( R \)-homomorphism \( f: M_1 \to M \) such that \( f \circ \alpha)(a) - a \in \rho(M) \), for all \( a \in M \). Thus Proposition 2.13. implies that \( M \) is a \( \rho \)-direct summand of \( M_1 \).

(2) \implies (3) and (3) \implies (4) are clear.

(4) \implies (1) Suppose that \( M \) is a \( \rho \)-direct summand of at least, one injective extension \( R \)-module of \( M \), say \( E \). To prove that \( M \) is a \( \rho \)-injective module. Consider the diagram (1) with exact row, where \( A \) and \( B \) are \( R \)-modules and \( f: A \to M \) is an \( R \)-homomorphism.

\[ \begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f & & \downarrow h \\
0 & \longrightarrow & B \\
\downarrow g & & \downarrow h \\
0 & \longrightarrow & M
\end{array} \] (diagram (1))

Since \( E \) is an extension of \( M \), there is an \( R \)-monomorphic, say \( \beta: M \to E \). Thus we have the diagram (2)

\[ \begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f & & \downarrow h \\
0 & \longrightarrow & B \\
\downarrow g & & \downarrow h \\
0 & \longrightarrow & M
\end{array} \] (diagram (2))
Since $E$ is an injective $R$-module, there exists an $R$-homomorphism $g: B \to E$ such that $(g \circ \alpha)(a) = (\beta \circ f)(a)$ for all $a$ in $A$. Thus we have the commutative diagram (3) with exact rows.

\[
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & B & \\
& & f & & g & \\
& & h & & & \\
0 & \longrightarrow & M & \longrightarrow & E & \\
\end{array}
\]

(dia}}{{m}}(3))

Since $M$ is a $\rho$-direct summand of $E$ (by hypothesis), thus there exists a homomorphism $h: B \to M$ such that $(h \circ \alpha)(a) - f(a) \in \rho(M)$, for all $a$ in $A$. Thus, for the diagram (1), we get a homomorphism $h: B \to M$ such that $(h \circ \alpha)(a) - f(a) \in \rho(M)$, for all $a$ in $A$. Therefore, $M$ is $\rho$-injective.

(3) $\Rightarrow$ (5) This is clear.

(5) $\Rightarrow$ (1) Suppose that $M$ is a $\rho$-direct summand of $E(M)$. Since $E(M)$ is an injective extension of $M$, thus $M$ is a $\rho$-direct summand of at least, one injective extension of itself. □

In the following corollary we will give an inner characterization of $\rho$-injective modules, for the term inner see [7].

**Corollary 2.15.** An $R$-module $M$ is $\rho$-injective if and only if $M$ is a $\rho$-direct summand of an $R$-module $\text{Hom}_R(R, B)$, with $B$ is a divisible Abelian group.

**Proof:** ($\Rightarrow$) Suppose that $M$ is $\rho$-injective. By [7, p.91], there is a $\mathbb{Z}$-monomorphism $f: M \to B$, where $B$ is a divisible Abelian group. Thus Lemma (5.5.2) in [7] implies that $\text{Hom}_R(R, B)$ is an injective $R$-module. Define $\theta: M \to \text{Hom}_R(R, B)$ by $\theta(m)(r) = f(rm)$, for all $m \in M$ and for all $r \in R$. It is easy to see that $\theta$ is an $R$-monomorphism and hence $\text{Hom}_R(R, B)$ is an extension $R$-module of $M$. Since $M$ is a $\rho$-injective $R$-module, thus Theorem 2.14. implies that $M$ is a $\rho$-direct summand of $\text{Hom}_R(R, B)$.

($\Leftarrow$) Suppose that $M$ is a $\rho$-direct summand of an $R$-module $\text{Hom}_R(R, B)$.

Now that $\text{Hom}_R(R, B)$ is an injective $R$-module. Thus $M$ is a $\rho$-direct summand of an injective extension $R$-module. Therefore, $M$ is a $\rho$-injective $R$-module, by Theorem 2.14. □

An $R$-monomorphism $\alpha: N \to M$ (where $N$ and $M$ are $R$-modules) is called split, if there exists an $R$-homomorphism $\beta: M \to N$ such that $\beta \circ \alpha = 1_N$ [7].

An $R$-module $M$ is injective if and only if for every $R$-module $N$, each $R$-monomorphism $\alpha: M \to N$ is split [7].

For analogous result for $\rho$-injective modules, we introduce the following concept.

**Definition 2.16.** An $R$-monomorphism $\alpha: N \to M$ is said to be $\rho$-split, if there exists an $R$-homomorphism $\beta: M \to N$ such that $(\beta \circ \alpha)(a) - a \in \rho(N)$, for all $a$ in $N$.

\[
\begin{array}{cccccc}
0 & \longrightarrow & N & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & 0 \\
& & I_N & \downarrow & & \\
& & & & & \\
\end{array}
\]

The following theorem gives and characterization of $\rho$-injectivity by using $\rho$-split monomorphisms.

**Theorem 2.17.** The following statements are equivalent for an $R$-module $M$:

(1) $M$ is $\rho$-injective;

(2) for each $R$-module $N$, each $R$-monomorphism $\alpha: M \to N$ is a $\rho$-split;

(3) for each injective $R$-module $N$, each $R$-monomorphism $\alpha: M \to N$ is a $\rho$-split;

(4) each $R$-monomorphism $\alpha: M \to E(M)$ is $\rho$-split.

**Proof:** (1) $\Rightarrow$ (2) Suppose that $M$ is a $\rho$-injective $R$-module. Let $N$ be any $R$-module and let $\alpha: M \to N$ be any $R$-monomorphism. Consider the following diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & M & \xrightarrow{\alpha} & N & \\
& & I_M & \downarrow & \beta & \\
& & & & M & \\
\end{array}
\]
Since $M$ is $\rho$-injective, there exists an $R$-homomorphism $\beta: N \to M$ such that 
$(\beta \circ \alpha)(a) - a \in \rho(M)$, for all $a \in M$. Hence $\alpha$ is a $\rho$-split.

$(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (1)$ Suppose that each $R$-monomorphism $\alpha: M \to E(M)$ is a $\rho$-split. To prove that $M$ is a $\rho$-injective. Consider the following diagram with exact row, where $A$ and $B$ are $R$-modules and $g: A \to M$ is any $R$-homomorphism.

```
0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} M
```

Since $E(M)$ is an extension of $M$, thus there is a monomorphism, say $\alpha: M \to E(M)$ and hence we get the following diagram with exact row.

```
0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} M \xrightarrow{h_1} E(M) \xrightarrow{h} M
```

Since $E(M)$ is an injective module, there exists a homomorphism $h: B \to E(M)$ such that 
$(h \circ f)(a) = (\alpha \circ g)(a)$, for all $a \in A$. By hypothesis, we have $\alpha: M \to E(M)$ is a $\rho$-split and hence there exists a homomorphism $\beta: E(M) \to M$ such that 
$(\beta \circ \alpha)(a) - a \in \rho(M)$, for all $a \in M$.

Put $h_1 = \beta \circ h$, it is clear that $h_1$ is an $R$-homomorphism. For each $a$ in $A$, we have that 
$(h_1 \circ f)(a) - g(a) = ((\beta \circ h) \circ f)(a) - g(a) = (\beta \circ f)(a) - g(a) = (\beta \circ \alpha)(g(a)) - g(a) \in \rho(M)$. Thus $(h_1 \circ f)(a) - (\beta \circ f)(a) - g(a) \in \rho(M)$, for all $a \in A$ and hence $M$ is a $\rho$-injective module. □

The following proposition gives a characterization of $\rho$-injective modules by using the class of injective modules.

**Proposition 2.18.** The following statements are equivalent for an $R$-modules $M$:

1. $M$ is $\rho$-injective;
2. $M$ is $\rho$-B-injective, for every injective module $B$;
3. for each diagram with $B$ is an injective $R$-module and $A$ is an essential submodule in $B$,

```
0 \rightarrow A \xrightarrow{i_A} B \xrightarrow{g} M \xrightarrow{f} B
```

there exists a homomorphism $g: B \to M$ such that 
$(g \circ i_A)(a) - f(a) \in \rho(M)$, for all $a \in A$.

**Proof:** $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (1)$ Consider the following diagram with $B$ is any $R$-module and $A$ is any essential submodule in $B$.

```
0 \rightarrow A \xrightarrow{i_A} B \xrightarrow{f} M \xrightarrow{h} E
```

By [1], there exists an injective $R$-module say $E$, such that $B$ is an essential submodule in $E$.

Thus we have the following diagram,

```
0 \rightarrow A \xrightarrow{i_A} B \xrightarrow{f} M \xrightarrow{g} B \xrightarrow{r} E
```

where $i_A$ and $i_b$ are inclusion $R$-homomorphisms. Since $A \leq E$ $B$ (by hypothesis) and $B \leq E$ it follows from [8] that $A \leq E$. By hypothesis, there exists an $R$-homomorphism $h: E \to M$ such that 
$(h \circ i_b \circ i_A)(a) - f(a) \in \rho(M)$, for all $a \in A$.

Put $g = h \circ i_b$, thus $(g \circ i_A)(a) = f(a) \in \rho(M)$. Since $A \leq E$ $B$ (by hypothesis) and $B \leq E$ it follows from [8] that $A \leq E$. By hypothesis, there exists an $R$-homomorphism $h: E \to M$ such that 
$(h \circ i_b \circ i_A)(a) - f(a) \in \rho(M)$, for all $a \in A$.

Put $g = h \circ i_b$, thus $(g \circ i_A)(a) = f(a) \in \rho(M)$.
\(\rho(M)\), for all \(a \in A\). By Proposition 2.4., \(M\) is \(\rho\)-B-injective, for every \(R\)-module \(B\) and hence \(M\) is a \(\rho\)-injective \(R\)-module. \(\square\)

In the following proposition, we will give another characterization of \(\rho\)-injectivity by using the class of free modules.

**Proposition 2.19.** An \(R\)-module \(M\) is \(\rho\)-injective if and only if \(M\) is \(\rho\)-\(F\)-injective, for every free \(R\)-module \(F\).

**Proof:** (\(\Rightarrow\)) This is obvious.

\((\Leftarrow\)) Suppose that \(M\) is \(\rho\)-\(F\)-injective, for every free \(R\)-module \(F\). Consider the following diagram with exact row.

\[
\begin{array}{ccc}
0 & \rightarrow & A & \xrightarrow{f} & B & \xrightarrow{i} & F \\
\downarrow g & & \downarrow h & & \downarrow h_1 \\
M & & \ast & & \ast & & \ast
\end{array}
\]

Since \(B\) is a set, thus there exists a free \(R\)-module, say \(F\), such that \(B\) is a basis of \(F\) [12, p.58]. By hypothesis, there exists an \(R\)-homomorphism \(h_1 : F \rightarrow M\) such that \((h_1 \circ (i \circ f))(a) - g(a) \in \rho(M)\), for all \(a \in A\). Put \(h := h_1 \circ i : B \rightarrow M\), it is clear that \(h\) is an \(R\)-homomorphism. For every \(a \in A\), we have that \((h \circ f)(a) - g(a) = ((h_1 \circ i \circ f)(a) - g(a)) \in \rho(M)\) and hence \(M\) is a \(\rho\)-injective \(R\)-module. \(\square\)

3. **Endomorphism Ring of \(\rho\)-Injective Modules**

Let \(M\) be an \(R\)-module, \(S = \text{End}_R(M)\) and let \(\Delta = \{ f \in S \mid \ker(f) \leq^e M \}\). It is well-known that \(\Delta\) is a two-sided ideal of \(S\) [13] and if an \(R\)-module \(M\) is injective, then the ring \(S/\Delta\) is regular. Moreover, if \(\Delta = 0\), then the ring \(S\) is a right self-injective ring [8].

For analogous results for \(\rho\)-injective modules we consider the following.

Let \(M\) and \(N\) be \(R\)-modules and \(f : M \rightarrow N\) be an \(R\)-homomorphism. The set \(f^{-1}(\rho(N)) = \{ x \in M \mid f(x) \in \rho(N) \}\) is said to be the kernel of \(f\) relative to a preradical \(\rho\) and denoted by \(\rho\ker(f)\).

Let \(M\) be an \(R\)-module and \(S = \text{End}_R(M)\). We will use the notation \(\rho\Delta\) for the set \(\{ f \in S \mid \rho\ker(f) \leq^e M \}\).

**Proposition 3.1.** Let \(M\) be an \(R\)-module and \(S = \text{End}_R(M)\). Then \(\rho\Delta\) is a two-sided ideal of \(S\).

**Proof.** Since the zero function belong to \(\Delta\), thus \(\rho\Delta\) is a non-empty set. Let \(f, g \in \rho\Delta\), thus \(\rho\ker(f) \leq^e M\) and \(\rho\ker(g) \leq^e M\) and hence Lemma 5.1.5(b) in [7] implies that \(\rho\ker(f) \cap \rho\ker(g) \leq^e M\). Since \(\rho\ker(f) \cap \rho\ker(g) \subseteq \rho\ker(f - g)\), thus \(\rho\ker(f - g) \leq^e M\) by [7, Lemma 5.1.5(a)] and hence \(f - g \in \rho\Delta\).

Let \(f \in \rho\Delta\) and \(h \in S\), thus \(\rho\ker(f) \leq^e M\). Since \(\rho\ker(f) \subseteq \rho\ker(h \circ f)\), thus \(\rho\ker(h \circ f) \leq^e M\) by [7, Lemma 5.1.5(a)] and hence \(h \circ f \in \rho\Delta\). Now we will prove that \(f \circ h \in \rho\Delta\). Since \(\rho\ker(f) \leq^e M\), then Lemma 5.1.5(c) in [7] implies that \(h^{-1}(\rho\ker(f)) \leq^e M\). But \(h^{-1}(\rho\ker(f)) \subseteq \rho\ker(f \circ h)\), therefore \(\rho\ker(f \circ h) \leq^e M\), by [7, Lemma 5.1.5(a)]. Thus \(f \circ h \in \rho\Delta\) and hence \(\rho\Delta\) is a two-sided ideal of \(S\). \(\square\)

Now, we are ready to state and prove the main result in this section.

**Theorem 3.2.** Let \(M\) be an \(R\)-module and \(S = \text{End}_R(M)\). If \(M\) is \(\rho\)-injective, then:

1. \(S/\rho\Delta\) is a regular ring;
2. if \(\rho\Delta = 0\), then \(S\) is a right self-injective ring.

**Proof.** Suppose that \(M\) is a \(\rho\)-injective \(R\)-module.

1. Let \(\lambda \in \rho\Delta \subseteq S/\rho\Delta\), thus \(\lambda \in \rho\Delta\). Put \(K = \ker(\lambda)\) and let \(L\) be a relative complement of \(K\) in \(M\). Define \(a : \lambda(L) \rightarrow M\) by \(a(\lambda(x)) =\)
x, for all \( x \in L \). It is easy to prove that \( \alpha \) is a well-defined \( R \)-homomorphism. Thus we have the following diagram, where \( i \) is the inclusion \( R \)-homomorphism.

\[
\begin{array}{ccc}
0 & \longrightarrow & L \\
& \alpha \downarrow & \beta \\
& M \\
\end{array}
\]

Since \( M \) is \( \rho \)-injective (by hypothesis), there exists an \( R \)-homomorphism \( \beta : M \rightarrow M \) such that \( \beta(\alpha(x)) - \alpha(\alpha(x)) \in \rho(M) \) for each \( x \in L \).

That is for each \( x \in L \), we have that

\[
\beta(\alpha(x)) = \alpha(\alpha(x)) + m_x,
\]

for some \( m_x \in \rho(M) \). Let \( u \in K \oplus L \), then \( u = x + y \) where \( x \in K \) and \( y \in L \) and hence \((\lambda - \lambda\beta\lambda)(u) = (\lambda - \lambda\beta\lambda)(x + y) = \lambda(x) + (\lambda - \lambda\beta\lambda)(y) + \lambda(y) = \lambda(x) + \lambda(y) - \lambda(\lambda\beta\lambda)(y)

\[
= 0 - \lambda(y) - \lambda(x) + \lambda(y) - \lambda(m_y) \in \rho(M) \text{ (because } \rho \text{ is a preradical) and hence } \lambda \in \rho(ker(\lambda - \lambda\beta\lambda)).
\]

Therefore, for each \( u \in K \oplus L \), we have that \( u \in \rho(ker(\lambda - \lambda\beta\lambda)) \). Since \( K \oplus L \leq M \) [8], thus Lemma 5.1.5(a) in [7] implies that \( \rho(ker(\lambda - \lambda\beta\lambda)) \leq M \) and hence \( \lambda - \lambda\beta\lambda \in \rho(M) \). Thus \( \lambda - \rho(M) = (\lambda\beta\lambda) + \rho(M) \) and hence \( S/\rho(M) \) is a regular ring.

(2) Suppose that \( \rho(M) = 0 \), thus by (1) above, we have that \( S \) is a regular ring. Let \( I \) be any right ideal of \( S \) and let \( f : I \rightarrow S \) be any right \( S \)-homomorphism. Consider the following diagram.

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
& \downarrow f & \rightarrow \\
& S \\
\end{array}
\]

Let \( IM \) be the \( R \)-submodule of \( M \) generated by \{ \lambda m : \lambda \in L, m \in M \}. Thus, if \( x \in IM \), then \( x = \sum_{i=1}^{n} \lambda_i m_i \) for some \( \lambda_1, \lambda_2, \ldots, \lambda_n \in L \) and some \( m_1, m_2, \ldots, m_n \in M \) where \( n \in \mathbb{Z}^+ \).

Define \( \theta : IM \rightarrow M \) as follows, for each \( x = \sum_{i=1}^{n} \lambda_i m_i \in IM \), put

\[
\theta(x) = \theta(\sum_{i=1}^{n} \lambda_i m_i) = \sum_{i=1}^{n} f(\lambda_i)(m_i).
\]

Let \( x, y \in IM \), thus \( x = \sum_{i=1}^{n} \lambda_i m_i \) and \( y = \sum_{j=1}^{m} \alpha_j m_j \), for some \( \lambda_i, \alpha_j \in L \) and \( m_i, m_j \in M \) with \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) where \( n, m \in \mathbb{Z}^+ \). Since \( S \) is a regular ring, thus Proposition 4.14 in [8] implies that each finitely generated right ideal of \( S \) is generated by an idempotent. Hence the right ideal of \( S \) which is generated by \( \lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_m \) written as \( eS \), where \( e = e^2 \in L \) and hence \( \lambda_1, \alpha_j \in eS \) for all \( i = 1, \ldots, n, j = 1, \ldots, m \) and this implies that \( \lambda_i = eh_i \) and \( \alpha_j = eh_j \) for some \( h_i, h_j \in S \) and for all \( i = 1, \ldots, n, j = 1, \ldots, m \). Hence \( e\lambda_i = e(h_i) = e^2 h_i = eh_i = \lambda_i \) for all \( i = 1, \ldots, n \) and \( e\alpha_j = e(h_j) = e^2 h_j = eh_j = \alpha_j \) for all \( j = 1, \ldots, m \). Thus, \( f(\lambda_i) = f(e)(\lambda_i) \) and \( f(\alpha_j) = f(e)(\alpha_j) \) for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). Therefore, \( \theta(x) = \theta(\sum_{i=1}^{n} \lambda_i m_i) = \sum_{i=1}^{n} f(\lambda_i)(m_i) = \sum_{i=1}^{n} f(\lambda_i) m_i = f(e) \sum_{i=1}^{n} \lambda_i m_i = f(e)x \) and similarly we have that \( \theta(y) = f(e)y \). Clearly, \( \theta \) is a well-defined \( R \)-homomorphism, since for all \( x, y \in IM \), if \( x = y \), then \( f(e)x = f(e)y \). Since \( \theta(x) = f(e)x \) and \( \theta(y) = f(e)y \) (as above), thus \( \theta(x) = \theta(y) \). Let \( x, y \in IM \) and \( r \in R \), thus \( \theta(x + y) = f(e)(x + y) = f(e)x + f(e)y = \theta(x) + \theta(y) \) and \( \theta(rx) = f(e)(rx) = r\theta(x) \). Therefore, \( \theta \) is a well-defined \( R \)-homomorphism. Thus we have the following diagram (where \( i \) is the inclusion \( R \)-homomorphism).

\[
\begin{array}{ccc}
0 & \longrightarrow & IM \\
& \downarrow \theta & \rightarrow \\
& M \\
\end{array}
\]

Since \( M \) is a \( \rho \)-injective, there exists an \( R \)-homomorphism \( \varphi : M \rightarrow M \) such that \( \varphi(x) - \theta(x) \in \rho(M) \), for all \( x \in IM \).

Let \( m \in M \) and \( \lambda \in L \). Thus \( (\varphi \lambda)(m) = \varphi(\lambda m) = \theta(\lambda m) + l_m = f(\lambda)(m) + l_m \), for some \( l_m \in \rho(M) \) and hence \( (\varphi \lambda - f(\lambda))(m) \).
Proposition 3.3. Let $M$ be an $\rho$-injective $R$-module and $S = \text{End}_R(M)$. Then
$I \cap K = IK + \rho \Delta \cap (I \cap K)$, for every two-sided ideals $I$ and $K$ of $S$.

Proof. Suppose that $M$ is a $\rho$-injective $R$-module, thus Theorem 3.2. implies that $S/\rho \Delta$ is a regular. Let $I$ and $K$ be any two-sided ideals of $S$. Let $a \in I \cap K$, thus $a + \rho \Delta \in S/\rho \Delta$. Since $S/\rho \Delta$ is a regular ring, thus there exists an element $\beta + \rho \Delta \in S/\rho \Delta$ such that $a + \rho \Delta = \beta a + \rho \Delta$ and hence $a - \beta a \in \rho \Delta$. Since $a - \beta a \in I \cap K$, thus $a - \beta a \in \rho \Delta \cap (I \cap K)$. Put $a_1 = a - \beta a$, thus $a = \beta a + a_1 \in IK + \rho \Delta \cap (I \cap K)$ and hence $I \cap K \subseteq IK + \rho \Delta \cap (I \cap K)$. Since $IK \subseteq I$ and $IK \subseteq K$, thus $IK \subseteq I \cap K$. Since $\rho \Delta \cap (I \cap K) \subseteq (I \cap K)$, thus $IK + \rho \Delta \cap (I \cap K) \subseteq I \cap K$. Therefore, $I \cap K = IK + \rho \Delta \cap (I \cap K)$. □

By applying Proposition 3.3, we have the following result.

Corollary 3.4. Let $M$ be a $\rho$-injective $R$-module, $S = \text{End}_R(M)$ and let $K$ be any two-sided ideal of $S$. Then $K = K^2 + (\rho \Delta \cap K)$

In [14], Osowski showed that, for an $R$-module $M$, if $Z(M) = 0$, then the Jacobson radical of the ring $S = \text{End}_R(M)$ is zero. Also, if $M$ is an injective $R$-module with $Z(M) = 0$, then the ring $S = \text{End}_R(M)$ is a right self-injective regular [8].

In the following, we will state and prove analogous results for $\rho$-injective modules. Firstly, we need the following lemma.

Lemma 3.5. Let $M$ be an $\rho$-module and $S = \text{End}_R(M)$. Then for each $\lambda \in S$ and for each $x \in M$ we have $[\rho(M) : \lambda(x)]_R = [\rho \text{ker}(\lambda) : x]_R$.

Proof. Let $\lambda \in S$ and $x \in M$. Thus if $r \in [\rho(M) : \lambda(x)]$, then $\lambda(x)r \in \rho(M)$ and hence $\lambda(x)r \in \rho(M)$ and this implies that $xr \in \rho \text{ker}(\lambda)$ and so $x \in [\rho \text{ker}(\lambda) : x]_R$. Therefore, $[\rho(M) : \lambda(x)]_R \subseteq [\rho \text{ker}(\lambda) : x]_R$ and by similar way we can prove $[\rho(M) : \lambda(x)]_R \subseteq [\rho \text{ker}(\lambda) : x]_R$. □

Let $M$ be an $R$-module. It is easy to prove that the set $\{m \in M| [\rho(M) : m]_R$ is an essential ideal in $R\}$ is a submodule of $M$. This submodule is said to be the $\rho$-singular submodule of $M$ and denoted by $\rho Z(M)$.

The following proposition is an analogous result of the Osowski's result [14].

Proposition 3.6. Let $M$ be an $R$-module and $S = \text{End}_R(M)$. If $\rho Z(M) = 0$, then $\rho \Delta = 0$.

Proof. Suppose that $\rho Z(M) = 0$ and let $a \in \rho \Delta$, thus $\rho \text{ker}(a) \leq^R M$ and hence [8, Lemma 3, p. 46] implies that $[\rho(M) : a(x)]_R \leq^R R$, for each $x \in M$. Since $[\rho(M) : a(x)]_R = [\rho \text{ker}(a) : x]_R$ (by Lemma 3.5.), thus $[\rho(M) : a(x)]_R \leq^R R$ and hence $a(x) \in \rho Z(M)$. Since $\rho Z(M) = 0$ (by hypothesis), thus $a(x) = 0$, for all $x$ in $M$ (i.e. $a = 0$) and hence $\rho \Delta = 0$. □

The following corollary (for $\rho$-injective modules) is analogous of the statement for injective modules [8].

Corollary 3.7. Let $M$ be a $\rho$-injective $R$-module and $S = \text{End}_R(M)$. If $\rho Z(M) = 0$, then $S$ is a right self-injective regular ring.

Proof. Suppose that $M$ is a $\rho$-injective module with $\rho Z(M) = 0$. Thus Proposition 3.6. implies that $\rho \Delta = 0$. Therefore, $S$ is a right self-injective regular ring, by Theorem 3.2. □
Corollary 3.8. If $R$ is a self $\rho$-injective ring and $\rho Z(R) = 0$, then $R$ is a right self-injective regular ring.

Proof. Since $R \cong \text{End}_R(R)$, thus the result follows from Corollary 3.7. \qed

Let $R$ be a ring and $x \in R$. Let $x_\lambda : R \to R$ be the mapping defined by $x_\lambda (r) = r\lambda x$, for all $\lambda \in R$. Then $x_\lambda$ is an $R$-homomorphism and $\text{End}_R(R) = \{x_\lambda | x \in R\}$ [8].

Lemma 3.9. Let $R$ be a ring and $S = \text{End}_R(R)$. Define $\alpha : R/\rho Z(R) \to S/\rho \Delta$ as follows:

$\alpha(x + \rho Z(R)) = x_\lambda + \rho \Delta$ for each $x \in R$. Then $\alpha$ is an $R$-isomorphism.

Proof. It is easy. \qed

The following proposition is an analogous result of the statement for self-injective rings [15].

Proposition 3.10. If $R$ is a self $\rho$-injective ring, then $R/\rho Z(R)$ is a regular ring.

Proof. Let $\alpha : R/\rho Z(R) \to S/\rho \Delta$ be the $R$-isomorphism as in Lemma 3.9, where $S = \text{End}_R(R)$. Let $x + \rho Z(R) \in R/\rho Z(R)$, thus $\alpha(x + \rho Z(R)) = x_\lambda + \rho \Delta \in S/\rho \Delta$. Since $R$ is a self $\rho$-injective ring, thus $S/\rho \Delta$ is a regular ring. (by Theorem 3.2.) and this implies that there exists an element $y_\lambda + \rho \Delta \in S/\rho \Delta$ such that $x_\lambda + \rho \Delta = x_\lambda y_\lambda x_\lambda + \rho \Delta = (xx_\lambda )\lambda + \rho \Delta$. Since $\alpha$ is an $R$-isomorphism, thus $\alpha^{-1}$ exists and $\alpha^{-1}(x_\lambda + \rho \Delta) = \alpha^{-1}((xx_\lambda )\lambda + \rho \Delta)$. Hence $x + \rho Z(R) = xx_\lambda + \rho Z(R) = (x + \rho Z(R)) \cdot (y + \rho Z(R)) \cdot (x + \rho Z(R))$. Since $\alpha^{-1}(y_\lambda + \rho \Delta) = y + \rho Z(R) \in R/\rho Z(R)$, thus we get an element $y + \rho Z(R) \in R/\rho Z(R)$ such that $x + \rho Z(R) = (x + \rho Z(R)) \cdot (y + \rho Z(R)) \cdot (x + \rho Z(R))$. Therefore, $R/\rho Z(R)$ is a regular ring. \qed

References:


الخلاصة

مفهوم الموديولات الإغمارية نسبة إلى جذر إبتدائي. تُعرف الموديولات الإغمارية نسبة إلى جذر إبتدائي. يوجد عدة تحقيقات عن تعميمات الموديولات الإغمارية مثل الموديولات الإغمارية تقربياً والموديولات الإغمارية الخاصة. العديد من التحسينات وخصائص الموديولات الإغمارية نسبة إلى جذر إبتدائي قد أعطيت. درسنا حلقات التماثلات الموديولية ذاتية للموديولات الإغمارية نسبة إلى جذر إبتدائي. نتائج هذا العمل تُوجد وتستند إلى النتائج الموجودة في المصادف.

الكلمات المفتاحية: الموديولات الإغمارية، الموديولات الإغمارية تقربياً، الجذر الإبتدائي، حلقات التماثلات الموديولية ذاتية.

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